REFINED SEIBERG-WITTEN INVARIANTS

STEFAN BAUER

In the past two decades, gauge theoretic methods became indispensable when considering manifolds in dimension four. Initially, research centred around the moduli spaces of Yang-Mills instantons. Simon Donaldson had introduced the instanton equations into the field. Using cohomological data of the corresponding moduli spaces, he defined invariants which could effectively distinguish differentiable structures on homeomorphic manifolds. Some years later, Nathan Seiberg and Ed Witten introduced the monopole equations. In a similar spirit as in Donaldson theory, cohomological data of the corresponding moduli spaces went into the definition of Seiberg–Witten invariants. These new invariants turned out to be far easier to compute, seemingly carrying the same information on differentiable structures as Donaldson's. His report [14] gives a glimpse of the wealth of insights in 4-manifold topology that could be extracted from these invariants.

However, there is more information in the monopole equations than is seen by the Seiberg-Witten invariants. The additional information is due to an interpretation of the monopole equations in terms of equivariant stable homotopy. The fact that certain partial differential equations admit a stable homotopy interpretation is not at all surprising. Indeed, this has been known for decades [37]. The good news is that in the case of the monopole equations it actually is possible to make effective use of this fact. The stable homotopy approach to the monopole equations does not only give a different view on known results, but also new insights.

This article is a mixture of a survey and a research article. It serves the multiple aims of introducing to this area of research, carefully outlining its foundations, presenting the known results in a unified framework and, last but not least, proving new results.

The new results concern various improvements to the definition of the refined invariants in [7]. Theorem 2.1, for example, specifies a class of nonlinear Fredholm maps between certain infinite dimensional manifolds and shows that the path connected components of the space of all such maps are naturally described by stable cohomotopy groups. This makes it possible to define the refined Seiberg-Witten invariant as the homotopy class of the monopole map in a precise way, clarifying a point left open in [7]. The proof also indicates how to avoid ad hoc arguments used in [6].

Another improvement is on the assumption $b^+ > b_1 + 1$, which had been necessary in [7] for a comparison with Seiberg-Witten invariants. The situation is now

summarized in Theorem 4.5. The relation to Seiberg-Witten invariants is clarified without any restriction on b^+ or b_1. This includes in particular the wall-crossing phenomenon in the $b^+ = 1$ case, which had been missing in [7], and the case $b^+ = 0$.

1. The monopole map

The main part in the story to be told is figured by the monopole map

$$\mu : \mathcal{A} \to \mathcal{C},$$

which is defined for a closed Riemannian 4-manifold X after fixing a K-orientation, or equivalently both an orientation in the usual sense and a $spin^c$-structure $\mathfrak{s}$. In addition, also a background $spin^c$-connection has to be fixed. The monopole map then is a fiber preserving map between infinite dimensional vector bundles over the torus

$$Pic^{\mathfrak{s}}(X) \cong H^1(X; \mathbb{R})/H^1(X; \mathbb{Z}).$$

The refined invariant by definition is the homotopy class of the monopole map in a sense to be made precise in the next chapter. This homotopy class does not depend on the chosen Riemannian metric or the chosen $spin^c$-connection as these choices are parametrized by connected (indeed contractible) spaces and so, indeed, becomes an invariant of the K-oriented differentiable manifold X (cf. 7.6).

Spinors are a main requisite in the definition of the monopole map. Let's start with the spinor group. The group $Spin^c(4)$ consists of those pairs (u^+, u^-) of unitary rank two transformations which have the same determinant. If Δ^+ and Δ^- denote the two dimensional unitary representations on which the respective factors act, then the $Spin^c(4)$-representation $\text{Hom}_{\mathbb{C}}(\Delta^+, \Delta^-)$ admits a real structure. The choice of a basis for the real part H in this representation leads to a surjection $Spin^c(4) \to SO(4)$ with kernel isomorphic to the group $\mathbb{T}$ of complex numbers of unit length. An element h of H has an adjoint h^* and acts on $\Delta = \Delta^+ \oplus \Delta^-$ via $h(\delta^+, \delta^-) = (-h^*(\delta^-), h(\delta^+))$. This action extends to an action of the Clifford algebra generated by H, resulting in an isomorphism $Cl(H) \otimes_{\mathbb{R}} \mathbb{C} \to \text{End}_{\mathbb{C}}(\Delta)$ of $Spin^c(4)$-representations. Combining with the complexified inverse to the isomorphism $Cl(H) \to \Lambda(H)$, which maps the product $h_1 h_2$ to $h_1 \wedge h_2 - \langle h_1, h_2 \rangle$, one obtains an isomorphism

$$\Lambda_{\mathbb{C}}(H) \to \text{End}_{\mathbb{C}}(\Delta)$$

of $Spin^c(4)$-representations. The decomposition $\Delta = \Delta^+ \oplus \Delta^-$ is preserved by elements of $\Lambda^2_{\mathbb{C}}(H)$. The kernel of the induced linear map

$$\rho : \Lambda^2_{\mathbb{C}}(H) \to \text{End}_{\mathbb{C}}(\Delta^+)$$

consists of the anti-selfdual part $\Lambda^-_{\mathbb{C}}(H)$, its image of the traceless endomorphisms. The map ρ preserves the real structure, mapping the real selfdual part $\Lambda^+(H)$ isomorphically to the traceless skew Hermitian endomorphisms of Δ^+.

We may globalize the above identifications of $Spin^c(4)$-representations to identifications of bundles by taking fibred products with a principal $Spin^c(4)$-bundle. Particularly interesting are such principal bundles which arise as $Spin^c(4)$-reductions of the othonormal oriented frame bundle on an oriented Riemannian four-manifold X. These are called $spin^c$-structures. In fact, the following data do characterize a

$spin^c$-structure: Rank two Hermitian vector bundles S^+ and S^-, together with isomorphisms $\det(S^+) \cong \det(S^-)$ and $T_{\mathbb{C}}^* X \to \text{H}om_{\mathbb{C}}(S^+, S^-)$ of Hermitian bundles, the latter isomorphism preserving the real structures. In fact, such $spin^c$-structures always exist (compare below). Taking tensor products with Hermitian line bundles results in a free and transitive action of $H^2(X; \mathbb{Z})$ on the set of all $spin^c$-structures.

There is an interpretation of $spin^c$-structures which is special to manifolds up to dimension 4: Choosing a $spin^c$-structure on X is equivalent to choosing a stably almost complex structure, i.e. an endomorphims I on the Whitney sum of the tangent bundle with a trivial rank two bundle over X satisfying $I^2 = -\text{id}$. This is because the natural map

$$BU \to BSpin^c$$

between the respective classifying spaces induces isomorphisms of homotopy groups up to dimension 5. By a theorem of Hirzebruch and Hopf [26], there always exists a stably almost complex structure on an oriented 4-manifold. If it comes from an (unstably) almost complex structure, then its second Chern class equals the Euler class of X. Using the equality $c_1^2 - 2c_2 = p_1$ of characteristic classes, we derive as a necessary condition for a stably almost complex structure to be almost complex that its first Chern class satisfies $c_1^2 = 3\,sign(X) + 2\,e(X)$. If X is connected, this condition is also sufficient [26]. In this case the integer

$$(1) \qquad k = \frac{c_1^2 - sign(X)}{4} - (b^+ - b_1 + 1)$$

thus measures, how far a stably almost complex structure is away from being almost complex. Here b_1 denotes the first Betti number of X and $b^+ = \frac{1}{2}(b_2 + sign(X))$ is the dimension of a maximal linear subspace of the second de Rham group of X on which the cup product pairing is positive definite.

After fixing a background $spin^c$-connection A, a $spin^c$-structure on X allows to define a Dirac operator

$$D_A : \Gamma(S^+) \to \Gamma(S^-)$$

mapping positive spinors, i.e. sections of the Hermitian vector bundle S^+, to negative spinors. The local model for the symbol of this operator over a point in X is obtained by identifying the cotangent space with the real part H of $\text{H}om_{\mathbb{C}}(\Delta^+, \Delta^-)$. At each point in X this symbol is the generator of Bott periodicity, so it provides a K-theory orientation class (compare [3]) for the manifold X. Indeed, any K-theory orientation of X uniquely arises this way. The Dirac operator is complex elliptic. Its index is given by

$$(2) \qquad \text{ind}_{\mathbb{C}}(D_A) = \frac{c_1^2 - sign(X)}{8}.$$

Now fix a $spin^c$-structure on the 4-manifold X, which from now on will be assumed to be connected unless explicitly stated differently. The gauge group $\mathcal{G} = map(X, \mathbb{T})$ acts on spinors via multiplication with $u : X \to \mathbb{T}$, on $spin^c$-connections via addition of udu^{-1}. The map sending a pair (A, ϕ) consisting of a $spin^c$-connection and a positive spinor to $D_A(\phi)$ is equivariant with respect to the gauge group. The action of the gauge group on the space of $spin^c$-connections is not free. However, restriction to the subgroup $\mathcal{G}_0$ consisting of functions which take value 1 at a chosen point in X results in a free action. In particular, the based

gauge group $\mathcal{G}_0$ acts freely on the affine linear space $A + i\ker(\mathrm{d})$, where d denotes the de Rham differential on one-forms on X, with quotient

$$Pic^{\mathfrak{s}}(X) \cong H^1(X;\mathbb{R})/H^1(X;\mathbb{Z}).$$

The gauge group acting trivially on forms, we obtain $\mathcal{G}$-spaces

$$\begin{aligned}
\widetilde{\mathcal{A}} &= (A + i\ker(\mathrm{d})) \times \left(\Gamma(S^+) \oplus H^0(X;\mathbb{R}) \oplus \Omega^1(X)\right) \\
\widetilde{\mathcal{C}} &= (A + i\ker(\mathrm{d})) \times \left(\Gamma(S^-) \oplus \Omega^0(X) \oplus H^1(X;\mathbb{R}) \oplus \Omega^+(X)\right)
\end{aligned}$$

consisting of $spin^c$-connections, spinors and forms on X.

Consider the map $\widetilde{\mu} : \widetilde{\mathcal{A}} \to \widetilde{\mathcal{C}}$ defined by

$$(3) \qquad (A',\, \phi,\, f,\, a) \mapsto \left(A',\, D_{A'}\phi + ia\phi,\, \mathrm{d}^*a + f,\, a_{harm},\, \mathrm{d}^+a + \sigma(\phi)\right).$$

Here $\sigma(\phi)$ denotes the trace free endomorphism $i(\phi \otimes \phi^* - \frac{1}{2}\|\phi\|^2\,\mathrm{id})$ of S^+, considered via the map ρ as a selfdual 2-form on X. Restricted to forms, the map is familiar from Hodge theory: It is linear, injective with cokernel the space $H^+(X;\mathbb{R})$ of harmonic selfdual two-forms on X. The map $\widetilde{\mu}$ is equivariant with respect to the action of $\mathcal{G}$. Dividing by the free action of the pointed gauge group we obtain the monopole map

$$\mu = \widetilde{\mu}/\mathcal{G}_0 : \mathcal{A} \to \mathcal{C}$$

as a fiber preserving map between the bundles $\mathcal{A} = \widetilde{\mathcal{A}}/\mathcal{G}_0$ and $\mathcal{C} = \widetilde{\mathcal{C}}/\mathcal{G}_0$ over $Pic^{\mathfrak{s}}(X)$. The preimage of the section $(A',\, 0,\, 0,\, 0,\, -F_{A'}^+)$ of $\mathcal{C}$, devided by the residual $\mathbb{T}$-action, is called the moduli space of monopoles.

For a fixed $k > 2$, consider the fiberwise L_k^2 Sobolev completion $\mathcal{A}_k$ and the fiberwise L_{k-1}^2 Sobolev completion $\mathcal{C}_{k-1}$ of $\mathcal{A}$ and $\mathcal{C}$. The monopole map extends to a continuous map $\mathcal{A}_k \to \mathcal{C}_{k-1}$ over $Pic^{\mathfrak{s}}(X)$, which will also be denoted by μ.

We will use the following properties of the monopole map:

1.1. It is $\mathbb{T}$-equivariant.

1.2. Fiberwise, it is the sum $\mu = l + c$ of a linear Fredholm map l and a nonlinear compact operator c.

1.3. Preimages of bounded sets are bounded.

Equivariance is immediate. The action is the residual action of the subgroup $\mathbb{T}$ of gauge transformations which are constant functions on X. This group acts by complex multiplication on the spaces $\Gamma(S^\pm)$ of sections of complex vector bundles and trivially on forms.

Restricted to a fiber, the monopole map is a sum of the linear Fredholm operator l, consisting of the elliptic operators D_A and $\mathrm{d}^*+\mathrm{d}^+$, complemented by projections to and inclusions of harmonic forms. The nonlinear part of μ is built from the bilinear terms $a\phi$ and $\sigma(\phi)$. Multiplication $\mathcal{A}_k \times \mathcal{A}_k \to \mathcal{C}_k$ is continuous for $k > 2$. Combined with the compact restriction map $\mathcal{C}_k \to \mathcal{C}_{k-1}$ we gain the claimed compactness for c: Images of bounded sets are contained in compact sets.

Compact perturbations $l + c : \mathcal{U}' \to \mathcal{U}$ of linear Fredholm maps between Hilbert spaces enjoy a nice topological property: The restriction to any bounded, closed subset is proper. The argument is straightforward: Let p denote a projection to the kernel of l. Then the restriction of $l+c$ to a closed subset $A \subset \mathcal{U}'$ factors through an

injective and closed and thus proper map $A \to \mathcal{U} \times \overline{c(A)} \times \overline{p(A)}$, $a \mapsto (l(a), c(a), p(a))$, a homeomorphism $(u, s, e) \mapsto (u + s, s, e)$ and the projection to $\mathcal{U}$, which is proper as the two other factors are compact.

If the bundles $\mathcal{A}$ and $\mathcal{C}$ were finite dimensional, then the boundedness property would be equivalent to properness. In this infinite dimensional setting, the argument above can be used the same way as Heine-Borel in the finite dimensional case to show that the boundedness condition implies properness. It turns out that the ingredients of the compactness proof for the moduli space [44] also prove the stronger boundedness property [7]: The Weitzenböck formula for the Dirac operator associated to the connection $A' = A + ib + ia$ reads

$$D_{A'}^* D_{A'} = \nabla_{A'}^* \nabla_{A'} + \frac{1}{4} s + F_{A'}^+.$$

Applying the Laplacian $\Delta |\phi|^2$ to the spinor part of an element $(A + ib, \phi, f, a)$ in the preimage of μ leads to an estimate

$$\Delta |\phi|^2 \leq 2 \langle D_{A'}^* D_{A'} \phi - \frac{s}{4} \phi - \frac{1}{2} F_{A'}^+ \phi, \ \phi \rangle.$$

The crucial point is that the term $F_{A'}^+$ can be replaced by an expression involving $\sigma(\phi)$ and terms which are straightforward to estimate. The Laplacian at the maximum is non-negative. Use of this fact and standard elliptic and Sobolev estimates then lead to an estimate

$$\|\phi\|_\infty^4 \leq P(\|\phi\|_\infty)$$

with a polynomial P of order 3. The boundedness property follows easily from this.

2. Enter stable homotopy

In case the first Betti number of X vanishes, the monopole map is a map between Hilbert spaces. The boundedness property (1.3) of μ is equivalent to the statement that μ extends continuously to a map $S^{\mathcal{A}} \to S^{\mathcal{C}}$ between the one-point completions, where the neighbourhoods of the points at infinity are the complements of bounded sets. As spaces, these one-point completions are infinite dimensional spheres. The monopole map thus rightly may be considered as a continuous map between spheres.

In the general case, we use a trivialisation $\mathcal{C} \cong Pic^{\mathfrak{s}}(X) \times \mathcal{U}$ of the bundle $\mathcal{C}$ to compose the monopole map with the projection p to the fiber $\mathcal{U}$. Now the boundedness property of μ translates as follows: The map $p \circ \mu$ extends continuously to a map $T\mathcal{A} \to S^{\mathcal{U}}$ from the Thom space of $\mathcal{A}$ to the sphere $S^{\mathcal{U}}$.

The idea of the refined invariant is to take the homotopy classes of these one-point completed maps. As it stands, this idea is of course nonsense: All the spaces involved are contractible, even equivariantly. So there is no interesting homotopy theory.

However, not all is lost. Restriction to maps satisfying not only (1.3), but also property (1.2) actually does the trick. We will consider the situation in a slightly more general setup.

Let $\mathcal{E}$ and $\mathcal{F}$ be infinite dimensional Hilbert space bundles over a compact base B. The structure group is the orthogonal group with its norm topology. Consider

the set $\mathcal{P}(\mathcal{E}, \mathcal{F})$ of fiber-preserving continuous maps $\phi : \mathcal{E} \to \mathcal{F}$ satisfying (1.2) and (1.3). Let's equip $\mathcal{P}(\mathcal{E}, \mathcal{F})$ with the topology induced by the metric

$$d(\phi, \psi) = \sup_{e \in \mathcal{E}} \|j\phi(e) - j\psi(e)\|,$$

where $j : \mathcal{F} \to \mathbb{R} \times \mathcal{F}$ denotes the embedding $f \mapsto (1 + f^2)^{-1}(1 - f^2, 2f)$ into the unit sphere bundle in $\mathbb{R} \times \mathcal{F}$ over B. (Actually, there are various topologies on $\mathcal{P}(\mathcal{E}, \mathcal{F})$ for which the following theorem is true; the choice made here is just to be definite.) Choosing a trivialisation $\mathcal{F} \cong B \times \mathcal{U}$ of the bundle $\mathcal{F}$, the path components of $\mathcal{P}(\mathcal{E}, \mathcal{F})$ roughly can be described through a bijection

$$\pi_0(\mathcal{P}(\mathcal{E}, \mathcal{F})) \cong \coprod_{\alpha \in KO(B)} \pi_{\mathcal{U}}^0(B; \alpha).$$

This description uses stable cohomotopy groups of B with "twisted coefficients". These groups need some explanation and as it stands, the statement is rather imprecise. "For the purposes of planning strategy" ([1]) it is useful, to think of this decomposition as presented over the group $KO(B)$. For the purpose of rigorous definitions and proofs, much more care has to be taken.

Let's start from the beginning, from pointed spaces. The prototype of a topological space with a distinguished base point, usually denoted by $*$, is the one-point compactification S^U of a finite dimensional real vector space U with the point at infinity as base point. The smash product $A \wedge C$ of pointed spaces is the quotient of their product obtained by identifying $A \times \{*\} \cup \{*\} \times C$ to a point. In this way $S^U \wedge S^V$ is canonically homeomorphic to $S^{U \oplus V}$. The sphere $S^{\mathbb{R}^n}$ is usually denoted by S^n. The smash product with S^1 induces a functor from pointed spaces to pointed spaces, called suspension.

According to Freudenthal's suspension theorem, which holds for finite dimensional spaces, iterated suspensions eventually induce isomorphisms of sets of pointed homotopy classes

$$[S^n \wedge A, S^n \wedge C] \to [S^{n+1} \wedge A, S^{n+1} \wedge C].$$

The notion of a spectrum arose from the desire to define a category in which the elements of the resulting abelian group

$$\operatorname*{colim}_{n \to \infty} [S^n \wedge A, S^n \wedge C]$$

appear as homotopy classes of maps between the objects. There are various ways to construct such categories. The situation suggests to use the Spanier-Whitehead category indexed by a universe: Objects and morphisms in this category are defined through colimit constructions. The index category consists of the finite dimensional linear subspaces of an infinite dimensional real Hilbert space $\mathcal{U}$, called universe, with inclusions as morphisms. So an object A in the Spanier-Whitehead category associates to $U \subset \mathcal{U}$ a pointed space A_U. To relate these spaces, we use the inclusion $U \subset W$ to identify W with $V \oplus U$, where V is the orthogonal complement to U in W. The collection of spaces A_U comes with identifications

(4) $$\sigma_{U,W} : S^V \wedge A_U \to A_W$$

satisfying the obvious compatibility condition

(5) $$\sigma_{U,W'} = \sigma_{W,W'} \circ (\mathrm{id}_{S^{V'}} \wedge \sigma_{U,W})$$

for $W' = V' \oplus W \subset \mathcal{U}$. The morphism set in the Spanier-Whitehead category is the colimit

$$\{A, C\}_{\mathcal{U}} = \operatorname*{colim}_{U \subset \mathcal{U}} [A_U, C_U].$$

over the maps

$$[A_U, C_U] \xrightarrow{\operatorname{id}_{S^V} \wedge \text{-}} [S^V \wedge A_U, S^V \wedge C_U] \leftrightarrow [A_W, C_W].$$

The latter identification is induced by the identifications $\sigma^A_{U,W}$ and $\sigma^C_{U,W}$.

Every (homotopy) category of spectra is supposed to contain some variant of the Spanier-Whitehead category as a full subcategory. So it should do no harm to call the objects spectra. It should, however, be pointed out that some authors reserve the name spectrum to objects in more elaborate categories.

Any pointed space A canonically defines its suspension spectrum, denoted by A as well, by setting $A_U = S^U \wedge A$.

To define objects in the Spanier-Whitehead category, it of course suffices to define them for a cofinal indexing category, as for example the subcategory of finite dimensional linear subspaces of $\mathcal{U}$ containing a fixed subspace U. So associating for a given pointed space A to $W = V \oplus U \subset \mathcal{U}$ the space $S^V \wedge A$ defines a spectrum different from A. We may safely denote this desuspension by $\Sigma^{-U} A$.

Let $p : \mathcal{F} \cong B \times \mathcal{U} \to \mathcal{U}$ be a trivialisation and suppose $l : \mathcal{E} \to \mathcal{F}$ is a continuous, fiberwise linear Fredholm map. Let $U \subset \mathcal{U}$ denote a finite dimensional linear subspace such that the index of l is represented by the difference $E - \underline{U}$ of finite dimensional vector bundles on B. Here $\underline{U}$ denotes the trivial vector bundle $p^{-1}(U)$ and $E = l^{-1}(\underline{U})$. The one-point compactification TE of E is called Thom space of E. The Thom spectrum is defined as $T(\operatorname{ind} l) = \Sigma^{-U} TE$. With this notation, stable cohomotopy with twisting $\operatorname{ind} l$ may be defined by

$$\pi^0_{\mathcal{U}}(B; \operatorname{ind} l) := \{T(\operatorname{ind} l), S^0\}_{\mathcal{U}}.$$

Such twisted cohomotopy groups are a natural habitat for Euler classes of vector bundles. To explain this, let F be a finite dimensional vector bundle over B. Choosing a bundle isomorphism $E \oplus F \cong \underline{U}$ and a section σ of F, this section and the projection to fibers together define a map $\sigma + \operatorname{id}_E$ extending continuously to one-point compactifications $TE \to S^U$. This map then represents the stable cohomotopy Euler class $e(F) \in \pi^0_{\mathcal{U}}(B; -F)$.

The relation to the Euler class of a bundle in a multiplicative cohomology theory h is as follows [8]: A Thom class $u \in h^r(B; -F) = h^r(TF, *)$ corresponds to an h-orientation of F. The h-theoretic Euler class is defined by $e_h(F) = \sigma^*(u) \in h^r(B)$. A generator $1 \in \tilde{h}^0(S^0)$ gives rise to the Hurewicz map $\pi^0(B; -F) \to h^0(B; -F)$, which associates to a stable pointed map $\phi : T(-F) \to S^0$ the element $h^0(\phi)(1)$. Using the product pairing $h^0(B; -F) \times h^r(B; F) \to \tilde{h}^r(B)$, the h-theoretic Euler class and the stable cohomotopy one are related by

$$e_h(F) = h^0(e(F))(1) \cdot u.$$

To formulate the theorem, let's introduce for a fixed fiberwise linear Fredholm operator $l : \mathcal{E} \to \mathcal{F}$ the subspace $\mathcal{P}_l(\mathcal{E}, \mathcal{F})$ of $\mathcal{P}(\mathcal{E}, \mathcal{F})$ consisting of elements ϕ such that $\phi - l$ is fiberwise compact.

Theorem 2.1. *A projection* $p : \mathcal{F} \cong B \times \mathcal{U} \to \mathcal{U}$ *induces a natural bijection*

$$\pi_0(\mathcal{P}_l(\mathcal{E}, \mathcal{F})) \cong \pi_{\mathcal{U}}^0(B; \operatorname{ind} l).$$

The theorem also handles homotopies by applying it to the base space $B \times [0, 1]$. Note that the restriction maps

$$\pi_{\mathcal{U}}^0(B \times [0, 1]; \operatorname{ind} l) \to \pi_{\mathcal{U}}^0(B \times \{i\}; \operatorname{ind} l|_{B \times \{i\}})$$

are isomorphisms. So if for example $\phi = l + c = l' + c'$ are two different presentations as a sum, then the constant homotopy $\phi = \phi_t = (1 - t)(l + c) + t(l' + c')$ can be used to identify $\pi_{\mathcal{U}}^0(B; \operatorname{ind} l)$ with $\pi_{\mathcal{U}}^0(B; \operatorname{ind} l')$. Under this identification, the element associated to the decomposition $l + c$ of ϕ is mapped to the element associated to the decomposition $l' + c'$.

Proof of 2.1. Let's briefly sketch a proof of the theorem: An element $\xi \in \pi_{\mathcal{U}}^0(B; \operatorname{ind} l)$ is represented by a virtual bundle $E - \underline{U}$ over B, together with a map $TE \to S^U$. It may be necessary to suspend the given map in order that it can be replaced by a homotopic map for which the preimage of the base point consists only of the base point. In particular, ξ then is represented by a proper map $E \to \underline{U}$. The given embedding of E into $\mathcal{E}$ and an identification of the orthogonal complements $E^\perp \subset \mathcal{E}$ and $\underline{U}^\perp \subset \mathcal{F}$, results in an element of $\mathcal{P}_l(\mathcal{E}, \mathcal{F})$.

On the other hand, for a given element $\phi \in \mathcal{P}_l(\mathcal{E}, \mathcal{F})$, choose a real number $R > 0$ and an ϵ with $0 < \epsilon < R$. The boundedness property (1.3) of ϕ implies that the preimage under $p\phi$ of the ball of radius R in $\mathcal{U}$ is bounded in $\mathcal{E}$. Using compactness of B, this bounded preimage is mapped by the fiberwise compact operator $p \circ (\phi - l)$ into a compact subset of $\mathcal{U}$. We may cover this image with finitely many ϵ-balls, the centers of which generate a finite dimensional vector space $U \subset \mathcal{U}$. After possibly enlarging U, we can assume that the virtual bundle $E - \underline{U}$ with $E = (pl)^{-1}(U)$ represents $\operatorname{ind} l$. The restriction $p\phi|_E$ by construction misses the sphere $S_R(U^\perp)$ of radius R in the orthogonal complement of $U \subset \mathcal{U}$. This map $p\phi|_E$ extends to the one-point completions to give a continuous map

$$TE \to S^{\mathcal{U}} \setminus S_R(U^\perp).$$

Composition with a homotopy inverse to the inclusion $S^U \to S^{\mathcal{U}} \setminus S_R(U^\perp)$ defines an element of $\{T(\operatorname{ind} l), S^0\}$. It remains of course to be checked that the two constructions lead to well defined maps between the sets in the theorem which are inverse to each other. This is straightforward, but a little tedious. Well-definedness uses the discussion in [7] and in particular lemma 2.3 there. The second construction obviously is left inverse to the first. To show that it is right inverse, one has to construct paths in $\mathcal{P}_l(\mathcal{E}, \mathcal{F})$ from an arbitrary element to an element, which can be "projected" onto the image of the first construction. Such a path is made explicit through the following homotopy ϕ_t which starts from $\phi = \phi_0$ and ends at ϕ_1. It is constant on a disk bundle of radius Q in $\mathcal{E}$, which contains the preimage of an R-disk bundle in $\mathcal{F}$. Outside it is defined by

$$(6) \qquad\qquad \phi_t(e) = \left(\frac{|e|}{Q}\right)^t \phi\left(\left(\frac{|e|}{Q}\right)^{-t} e\right). \quad \square$$

The theorem describes the path-connected components of these mapping spaces in terms of a disjoint union of algebraic objects. So one can hardly expect the algebraic

structure to be reflected in the world of Fredholm maps by natural constructions. In particular, addition of elements in the respective cohomotopy groups seems to be difficult to describe in terms of Fredholm maps without making use of the theorem. However, two aspects are inherent in the Fredholm setup.

Remarks:
2.2. A Fredholm map $\phi \in \mathcal{P}_l(\mathcal{E}, \mathcal{F})$, for which $p\phi$ is not surjective, describes the zero element in the stable cohomotopy group associated to its linearization l. To see this, recall from the previous section that ϕ is proper. In particular, the image of $p\phi$ is a closed subset in U. If a point $u \in \mathcal{U}$ is in the complement of the image, then so is a whole ϵ-neighbourhood of u. Now apply the construction above for some $R > |u| + \epsilon$ and replace ϕ by the homotopic ϕ_1. We choose U so that it also contains u. The map $p\phi_1|_E$, followed by the orthogonal projection to U is proper and by construction misses u. So its one-point compactification is null homotopic.

2.3. The bijection (2.1) respects products: If $\mathcal{E}_i$ and $\mathcal{F}_i$ denote Hilbert space bundles over a base space B_i, then taking products of maps results in a product

$$\mathcal{P}_{l_1}(\mathcal{E}_1, \mathcal{F}_1) \times \mathcal{P}_{l_2}(\mathcal{E}_2, \mathcal{F}_2) \to \mathcal{P}_l(\mathcal{E}_1 \times \mathcal{E}_2, \mathcal{F}_1 \times \mathcal{F}_2)$$

with $l = \mathrm{pr}_1^* l_1 \times \mathrm{pr}_2^* l_2$. This product structure is reflected in the stable cohomotopy counterpart. The natural smash product of objects in the Spanier-Whitehead category is not defined within one universe, but comes with a change of universes: Let A and C be objects in the Spanier-Whitehead categories indexed by universes $\mathcal{U}$ and $\mathcal{V}$. The smash product $A \wedge C$ then is an object in the Spanier-Whitehead category indexed by $\mathcal{U} \oplus \mathcal{V}$. It is defined by

$$(A \wedge C)_{U \oplus V} := A_U \wedge C_V,$$

whenever both sides make sense, which is at least for a cofinal subcategory of the indexing category.

If Fredholm maps $\phi_i \in \mathcal{P}_{l_i}(\mathcal{E}_i, \mathcal{F}_i)$ represent elements $\xi_i \in \pi^0_{\mathcal{U}_i}(B_i; \mathrm{ind}\, l_i)$, then the product $\phi_1 \times \phi_2$ represents the cohomotopy class $\xi_1 \wedge \xi_2$, which is an element of the group $\pi^0_{\mathcal{U}_1 \oplus \mathcal{U}_2}(B_1 \times B_2; \mathrm{ind}\, l)$.

3. Some equivariant topology

Using equivariant spaces and maps throughout, the above concepts carry over to an equivariant setting in a straightforward manner. An appropriate reference is [1].

The action of a compact Lie group G on a pointed G-space A fixes the distinguished base point. If A and C are pointed G-spaces, then the smash product $A \wedge C$ obtains a G-action by restricting the natural G $\times$ G-action to the diagonal subgroup. A G-universe $\mathcal{U}$ is a Hilbert space on which G acts via isometries in such a way that an irreducible G-representation, if contained in $\mathcal{U}$, is so with infinite multiplicity. A complete G-universe contains all irreducible representations.

An object of the G-Spanier-Whitehead category indexed by $\mathcal{U}$ associates to a finite dimensional representation $U \subset \mathcal{U}$ a pointed G-space A_U. The morphism set is the colimit

$$\{A, C\}^G_{\mathcal{U}} = \operatorname*{colim}_{U \subset \mathcal{U}} [A_U, C_U]^G$$

of G-homotopy classes of equivariant maps. This morphism set is a group if the G-universe $\mathcal{U}$ contains trivial G-representations.

An equivariant projection $p : \mathcal{F} \cong B \times \mathcal{U} \to \mathcal{U}$ need not exist. This may happen for example, if the G-action on the (unpointed) base space B of the G-Hilbert bundle $\mathcal{F}$ is nontrivial or if the fiber over a fixed point in B does not qualify as a universe. If such a projection p exists, it induces a natural bijection

$$\pi_0(\mathcal{P}_l(\mathcal{E}, \mathcal{F})^{\mathrm{G}}) \cong \pi^0_{\mathrm{G},\mathcal{U}}(B; \mathrm{ind}\, l)$$

as before.

The stable cohomotopy groups in 2.1 and in particular their equivariant counterparts are barely known. To get a rough impression, consider the case of a point $B = \{P\}$. If we choose a universe with a trivial G-action, then the twist $\mathrm{ind}\, l$ is characterized by an integer i and the group $\pi^0_{\mathrm{G},\mathcal{U}}(P; \mathrm{ind}\, l)$ can be identified with the i-th stable stem $\pi^{st}_i(S^0)$. On the other extreme, if we choose $\mathcal{U}$ to be a complete universe, then the isomorphism class of $\mathrm{ind}\, l$ gives an element in the representation ring $RO(\mathrm{G})$. In the case where l is an isomorphism, $\pi^0_{\mathrm{G},\mathcal{U}}(P; 0)$ is isomorphic to the Burnside ring $A(\mathrm{G})$ ([10], II.8.4). If G is a finite group, $A(G)$ is the Grothendieck ring of finite G-sets with addition given by disjoint union and multiplication given by product. A point with the trivial G-action on it represents 1.

Understanding the group $\pi^0_{\mathrm{G},\mathcal{U}}(P; \mathrm{ind}\, l)$ for a virtual representation $\mathrm{ind}\, l = V - W$ of G in any universe $\mathcal{U}$ whatsoever boils down to understanding the homotopy classes of G-maps $f : S^V \to S^W$. Equivariant K-theory provides some information in case both V and W are complex representations. The method is explained in [10], II.5:

Let $K_{\mathrm{G}}(B)$ be the Grothendieck group of equivariant complex vector bundles over the G-space B. For a pointed space B as usual $\tilde{K}_{\mathrm{G}}(B)$ denotes the kernel of the restriction homomorphism $K_{\mathrm{G}}(B) \to K_{\mathrm{G}}(*) \cong R(\mathrm{G})$. If V is a complex G-representation, then $K_{\mathrm{G}}(V) := \tilde{K}_{\mathrm{G}}(S^V)$ is a free $R(\mathrm{G})$-module generated by a Bott class $b(V)$ [2]. The image of the Bott class $b(U \oplus V)$ under the restriction homomorphism $\tilde{K}_{\mathrm{G}}(S^{U \oplus V}) \to K_{\mathrm{G}}(S^U)$ is $e_{K_{\mathrm{G}}}(V)b(U)$. This defines the Euler class, which was determined by Segal [38] to be the element

$$(7) \qquad\qquad e_{K_{\mathrm{G}}}(V) = \sum_{i=0}^{dimV} (-1)^i \Lambda^i(V) \in R(\mathrm{G}).$$

A pointed G-map $f : S^V \to S^W$ induces a $R(\mathrm{G})$-linear homomorphism in K_{G}-theory. The image of the Bott class of W is a multiple $a_{\mathrm{G}}(f)b(V)$ of the Bott class of V. The K_{G}-theory degree $a_{\mathrm{G}}(f)$ is an element of the complex representation ring $R(\mathrm{G})$.

To determine the K_{G}-degree as a character on G, we have to evaluate it at elements $g \in \mathrm{G}$. Let C denote the closure of the subgroup generated by g. Decompose $V = V_{\mathrm{C}} \oplus V^{\mathrm{C}}$ into the C-fixed point set V^{C} and its orthogonal complement. The inclusions of the fixed point sets induce a commuting diagram

$$\begin{array}{ccc} K_{\mathrm{C}}(W) & \xrightarrow{f^*} & K_{\mathrm{C}}(V) \\ \downarrow & & \downarrow \\ K_{\mathrm{C}}(W^{\mathrm{C}}) & \xrightarrow{f^{\mathrm{C}*}} & K_{\mathrm{C}}(V^{\mathrm{C}}). \end{array}$$

The lower map is multiplication by the degree of the map f^{C} as a map between oriented spheres, if V^{C} and W^{C} both have the same dimension. Otherwise it is zero. This is because $\tilde{K}(S^{2n}) \cong \mathbb{Z}$ and $\pi^{st}_{2n-2m}(S^0)$ is torsion for $n \neq m$.

Commutativity of the diagram relates Euler classes and degrees by

$$(8) \qquad e_{K_{\mathrm{C}}}(W_{\mathrm{C}})d(f^{\mathrm{C}}) = a_{\mathrm{C}}(f)e_{K_{\mathrm{C}}}(V_{\mathrm{C}}).$$

The representation V_{C} does not contain a trivial summand. So the character $e_{K_{\mathrm{C}}}(V_{\mathrm{C}})$ does not vanish at g. In particular, the K_{G}-degree can be computed from the ordinary degrees of the restrictions of f to fixed points.

Let's apply this concept to a simple example: Consider the group $\mathbb{T}$ of complex numbers of unit length acting on the representation V with character $z \mapsto n + mz$,

Proposition 3.1. *Let* $f : S^{2n} \wedge S^{\mathbb{C}^m} \to S^{2n} \wedge S^{\mathbb{C}^{m+l}}$ *be a* $\mathbb{T}$ *equivariant map such that the restricted map on the fixed points has degree* $d \neq 0$. *Then* $l \geq 0$ *and in case* $l = 0$ *the degree of* f *nonequivariantly on the total space is* d *as well.*

Proof: For $z \neq 1$, the above equation (8) reads as follows:

$$(1 - z)^{m+l}d = a_{\mathbb{T}}(f)(z)(1 - z)^m.$$

The function $d(1 - z)^l = a_{\mathbb{T}}(f)(z)$ is a character in $R(\mathrm{G})$ only if $l \geq 0$. In case $l = 0$, the $K_{\mathbb{T}}$-degree and hence the $(K\text{-})$degree equals d. $\qquad\square$

Algebraic topology provides quite heavy machinery for the equivariant world. Basic equipment can be found in [1], [10]. Here is a survival kit:

3.2. An equivariant isometry $\mathcal{U} \hookrightarrow \mathcal{V}$ induces a change-of-universe morphism $\{\_, \_\}^G_{\mathcal{U}} \to \{\_, \_\}^G_{\mathcal{V}}$. It is bijective, if both universes are built from the same irreducible representations.

3.3. A cofiber sequence $A' \to A \to A''$ of pointed G-spaces induces long exact seqences

$$\ldots \leftarrow \{S^i \wedge A', C\}^{\mathrm{G}}_{\mathcal{U}} \leftarrow \{S^i \wedge A, C\}^{\mathrm{G}}_{\mathcal{U}} \leftarrow \{S^i \wedge A'', C\}^{\mathrm{G}}_{\mathcal{U}} \leftarrow \{S^{i+1} \wedge A', C\}^{\mathrm{G}}_{\mathcal{U}} \leftarrow \ldots$$

$$\ldots \to \{C, S^i \wedge A'\}^{\mathrm{G}}_{\mathcal{U}} \to \{C, S^i \wedge A\}^{\mathrm{G}}_{\mathcal{U}} \to \{C, S^i \wedge A''\}^{\mathrm{G}}_{\mathcal{U}} \to \{C, S^{i+1} \wedge A'\}^{\mathrm{G}}_{\mathcal{U}} \to \ldots$$

3.4. Let $\mathrm{H} < \mathrm{G}$ be a subgroup of finite index. Then there are natural bijections

$$\{A, \mathrm{res}^{\mathrm{G}}_{\mathrm{H}}C\}^{\mathrm{H}}_{\mathrm{res}^{\mathrm{G}}_{\mathrm{H}}\mathcal{U}} \leftrightarrow \{(\mathrm{G} \sqcup *) \wedge_{\mathrm{H}} A, C\}^{\mathrm{G}}_{\mathcal{U}}.$$

$$\{\mathrm{res}^{\mathrm{G}}_{\mathrm{H}}C, A\}^{\mathrm{H}}_{\mathrm{res}^{\mathrm{G}}_{\mathrm{H}}\mathcal{U}} \leftrightarrow \{C, (\mathrm{G} \sqcup *) \wedge_{\mathrm{H}} A\}^{\mathrm{G}}_{\mathcal{U}}.$$

Here $\mathrm{G} \sqcup *$ denotes the group G with a disjoint base point and $(\mathrm{G} \sqcup *) \wedge_{\mathrm{H}} A$ is the orbit space of $(\mathrm{G} \sqcup *) \wedge A$ by the action $(h, (g, a)) \mapsto (gh^{-1}, ha)$. Here A is a Spanier-Whitehead spectrum for the group H and C one for the group G. The first adjointness property follows from a corresponding property on the space level. The second Wirthmüller isomorphism can be found in [10], II.6.14.

3.5. For spaces with free G-action, the equivariant cohomotopy is naturally isomorphic to the non-equivariant cohomotopy of the quotient space.

In more exact diction, this reads: Let A be finite dimensional G-space with a free G-action away from the base point and let C be a non-equivariant spectrum, indexed

by the fixed point universe $\mathcal{U}^{\mathrm{G}}$ of the universe $\mathcal{U}$ indexing the suspension spectrum of A. For such objects, there is a natural bijection

$$\{A, j^*C\}_{\mathcal{U}}^{\mathrm{G}} \leftrightarrow \{A/\mathrm{G}, C\}_{\mathcal{U}^{\mathrm{G}}}.$$

Here j^*C denotes the spectrum obtained from C, considered as a spectrum with trivial G-action, by change of universe $\mathcal{U}^{\mathrm{G}} \hookrightarrow \mathcal{U}$. This property is obvious for spaces. The fact that it carries over to the equivariantly stable world follows from a careful analysis of the equivariant suspension theorem: In this situation, it suffices to suspend with trivial representations only to get into the stable range.

4. Topology of the monopole map

The first statement in the following theorem summarizes the previous discussion.

Theorem 4.1. ([7]) *The monopole map $\mu : \mathcal{A} \to \mathcal{C}$ defines an element in the equivariant stable cohomotopy group*

$$\pi_{\mathbb{T},\mathcal{U}}^0 \left(Pic^{\mathfrak{s}}(X); \mathrm{ind}(D) - \underline{H^+(X;\mathbb{R})} \right).$$

For $b^+ > \dim(Pic^{\mathfrak{s}}(X)) + 1$, a homology orientation determines a homomorphism of this stable cohomotopy group to $\mathbb{Z}$, which maps $[\mu]$ to the integer valued Seiberg-Witten invariant.

The universe in this statement is explicitly given as the fiber

$$\mathcal{U} = \Gamma(S^-) \oplus \Omega^0(X) \oplus H^1(X;\mathbb{R}) \oplus \Omega^+(X)$$

of the bundle $\mathcal{C}$. The index of the linearization of the monopole map consists of two summands. The Dirac operator associated to $\mathfrak{s}$ defines a virtual complex index bundle $\mathrm{ind}(D)$ over the Picard torus. The second bundle is the trivial bundle with fiber the b^+-dimensional space of self-dual harmonic forms $H^+(X;\mathbb{R})$. An orientation of $Pic^{\mathfrak{s}}(X) \times H^+(X;\mathbb{R})$ is called a homology orientation.

Let's define the homomorphism of the stable cohomotopy group to $\mathbb{Z}$. An element of the stable cohomotopy group is represented by an equivariant map $f : TE \to S^U$ from the Thom space of a bundle E over $Pic^{\mathfrak{s}}(X)$, where $E - \underline{U} = \mathrm{ind}\, l$ is the index of the linearization l of μ. Let Ci denote the mapping cone of the inclusion $i : TE^{\mathbb{T}} \to TE$ of the $\mathbb{T}$-fixed point set. In the long exact sequence associated to the cofiber sequence $TE^{\mathbb{T}} \to TE \to Ci$,

(9) $\quad \pi_{\mathbb{T},\mathcal{U}}^{-1}(\Sigma^{-U}(TE^{\mathbb{T}})) \to \pi_{\mathbb{T},\mathcal{U}}^0(\Sigma^{-U}Ci) \to \pi_{\mathbb{T},\mathcal{U}}^0(\Sigma^{-U}TE) \to \pi_{\mathbb{T},\mathcal{U}}^0(\Sigma^{-U}(TE^{\mathbb{T}}))$

the first and last term are vanishing because by assumption the dimension of the space $S^1 \wedge TE^{\mathbb{T}}$ is less than the dimension of the $\mathbb{T}$-fixed point sphere in S^U, the difference in dimension being $b^+ - b_1 - 1$. So the map μ can be described by a cohomotopy element of $\Sigma^{-U}Ci$. The Hurewicz image $h(\mu)$ of this element in equivariant Borel-cohomology lies in the relative group $H_{\mathbb{T}}^0(\Sigma^{-U}TE, \Sigma^{-U}TE^{\mathbb{T}})$. The $\mathbb{T}$-action on the pair of spaces $(TE, TE^{\mathbb{T}})$ is relatively free. So its equivariant cohomology group identifies with the singular cohomology $H^*(TE/\mathbb{T}, TE^{\mathbb{T}})$ of the quotient. After replacing $TE^{\mathbb{T}}$ by a tubular neighbourhood, this is the singular cohomology of a connected manifold relative to its boundary. An orientation of $Pic^{\mathfrak{s}}(X)$ together with the standard orientation of complex vector bundles defines an orientation class $[TE]_{\mathbb{T}}$ in the top cohomology of this manifold. Similarly, the

chosen homology orientation of X and the orientation of $Pic^{\mathfrak{s}}(X)$ determine the orientation of U and thus a generator $\Sigma^{-U}[TE]_{\mathbb{T}}$ of the graded cohomology group $H^*_{\mathbb{T}}(\Sigma^{-U}TE, \Sigma^{-U}TE^{\mathbb{T}})$ in its top grading $* = k$. This cohomology group is a graded module over the polynomial ring $H^*_{\mathbb{T}}(*) \cong \mathbb{Z}[t]$ in one variable t of degree 2. The homomorphism sought for is zero if k is odd or negative. Otherwise $t^{\frac{k}{2}}h(\mu)$ is a multiple of the generator $\Sigma^{-U}[TE]_{\mathbb{T}}$. This multiplicity is the Seiberg-Witten invariant. $\qquad\square$

To see what happens in the cases $b^+ \leq b_1 + 1$ not covered by this theorem we have to take a closer look at the monopole map and distinguish different cases.

4.1. **The case $b^+ = 0$.** The choice of a point $P \in Pic^{\mathfrak{s}}(X)$ induces a restriction map

$$\pi^0_{\mathbb{T},\mathcal{U}}(Pic^{\mathfrak{s}}(X); \operatorname{ind} l) \to \pi^0_{\mathbb{T},\mathcal{U}}(P; \operatorname{ind} l) \cong \{S^{\operatorname{ind}(D)}, S^{H^+(X;\mathbb{R})}\}^{\mathbb{T}}_{\mathcal{U}}$$

The index of the Dirac operator $\operatorname{ind}(D)$ consists of $d = \frac{1}{8}(c^2 - \operatorname{sign}(X))$ copies of the tautological complex $\mathbb{T}$-representation. The restriction to the $\mathbb{T}$-fixed point set of an element in this group is an element in the stable stem $\pi^{st}_{-b^+}(S^0)$, which is trivial except in the case $b^+ = 0$. In this case the restriction of the monopole map is a linear isomorphism on the fixed point set. Here is an immediate consequence, well known from Seiberg-Witten theory:

Proposition 4.2. *Let X be an oriented 4-manifold with $b^+ = 0$. Then the first Chern class of any K-orientation on X satisfies $c^2 \leq \operatorname{sign}(X)$.*

Otherwise the monopole map represented an element in $\{S^{\mathbb{C}^d}, S^0\}^{\mathbb{T}}_{\mathcal{U}}$ for some $d > 0$ which is of degree 1 on the $\mathbb{T}$-fixed point set. The existence of such an element contradicts 3.1. $\qquad\square$

Applying Elkies' theorem [17], we obtain as a corollary Donaldson's theorem:

Theorem 4.3. ([11], [12]) *Let X be a closed oriented four-manifold with negative definite intersection form. Then the intersection pairing on $H_2(X;\mathbb{Z})/Torsion$ is diagonal.* $\qquad\square$

4.2. **The case $b^+ = 1, b_1 = 0$.** In the case $b^+ = 1$, the Seiberg-Witten invariants depend in a well understood manner ([44], [30]) on the Riemannian metric and on an additional perturbation parameter. To understand the phenomenon, let's illustrate it in a characteristic example. This example describes the situation in the case of an almost complex manifold with $b_1(X) = 0$, cf. [7]:

View the spinning globe as a two-sphere with an $\mathbb{T}$-action and choose the north pole as a base point. As a target space, take a one-sphere with trivial action and choose two points on this one-sphere as "poles", the north pole again as base point. Based equivariant maps from the spinning globe to the one-sphere are determined by their restriction to a latitude, which as an arc is a contractible space. So there is only the trivial homotopy class of equivariant such maps.

In contrast, consider equivariant maps, which take north and south pole to north and south pole, respectively. The monopole maps for all choices of metrics and background connections actually are of this type. Such a map basically wraps a latitude $n + \frac{1}{2}$ times around the one-sphere. Choosing a generic point in the one-sphere, the oriented count of preimages in a fixed latitude defines in a natural way

a map of the set of relative homotopy classes to the integers. This oriented count, however, depends on the choice of the generic point. It changes by ± 1, if the generic point is chosen in the "other half" of the one-sphere.

There are two ways to deal with the problem. If one prefers to have homotopy classes, one may consider the monopole map up to equivariant homotopy relative to the fixed point. The monopole map then is a well defined element in the set of all such homotopy classes. However, this set will not be a group anymore. There is a comparison map to the integers depending upon the choice of a "chamber". An alternative is described below.

4.3. The case $b^+ > 1, b_1 \neq 0$. The restriction of the monopole map to the $\mathbb{T}$-fixed point set $\mathcal{A}^{\mathbb{T}} = Pic^{\mathfrak{s}}(X) \times (H^0(X; \mathbb{R}) \oplus \Omega^1(X))$ is a product map. On the factor $Pic^{\mathfrak{s}}(X)$ it is the identity, on the second factor it is a linear embedding with cokernel $H^+(X; \mathbb{R})$. To take this into account, we would like to construct an equivariant spectrum Q encoding this information. Ideally, this spectrum would be obtained by a push-out of two maps. The one map describes the inclusion $\mathcal{A}^{\mathbb{T}} \to \mathcal{A}$ of the fixed point set, the other the projection $\mathcal{A}^{\mathbb{T}} \to H^0(X; \mathbb{R}) \oplus \Omega^1(X)$. Such a push-out seems not available in the category we are working in. Let's try to define a substitute. Suppose $E - \underline{U}$ for $U \subset \Gamma(S^-)$ represents the index of the Dirac operator as a virtual complex bundle over $Pic^{\mathfrak{s}}(X)$. Then let $TE/Pic^{\mathfrak{s}}(X)$ denote the quotient of the Thom space TE, where the subspace $Pic^{\mathfrak{s}}(X)$, the image of the zero section in E, is identified to a point. Alternatively, $TE/Pic^{\mathfrak{s}}(X)$ is described as the unreduced suspension of the unit sphere bundle in E. As a $\mathbb{T}$-space it has two fixed points. The spectrum $Q(X, \mathfrak{s}, U)$ then is defined by

$$Q(X, \mathfrak{s}, U) = \Sigma^{-U - H^+(X; \mathbb{R})} \left(TE/Pic^{\mathfrak{s}}(X) \right).$$

Using this spectrum, we obtain a straightforward sharpening of 4.1:

Proposition 4.4. *For sufficiently large $U \subset \Gamma(S^-)$, the monopole map $\mu : \mathcal{A} \to \mathcal{C}$ defines an element in the equivariant stable cohomotopy group $\pi^0_{\mathbb{T}, \mathcal{U}}(Q(X, \mathfrak{s}, U))$. For $b^+ > 1$, a homology orientation determines a homomorphism of this stable cohomotopy group to $\mathbb{Z}$, which maps $[\mu]$ to the integer valued Seiberg-Witten invariant.*

The proof is a slight variation of that of 4.1. To construct the homomorphism to $\mathbb{Z}$, use the cofiber sequence $S^0 \to TE/Pic^{\mathfrak{s}}(X) \to Ci$ of spaces. The outer terms in the analogue of sequence (9) now are vanishing for $b^+ > 1$ for dimension reasons. $\square$

One can prove that for U big enough the groups $\pi^0_{\mathbb{T}, \mathcal{U}}(Q(X, \mathfrak{s}, U))$ become isomorphic. The description, however, still doesn't look satisfactory.

4.4. The case $b^+ = 1, b_1 \neq 0$. Consider the cofiber sequence $S^0 \to TE/Pic^{\mathfrak{s}}(X) \to Ci$ in the proof of 4.4 and set $W = U + H^+(X; \mathbb{R}) \subset \mathcal{U}$. This leads to the analogue of the exact sequence (9)

$$\pi^{-1}_{\mathbb{T}, \mathcal{U}}(\Sigma^{-W} S^0) \to \pi^0_{\mathbb{T}, \mathcal{U}}(\Sigma^{-W} Ci) \to \pi^0_{\mathbb{T}, \mathcal{U}}(Q(U)) \to \pi^0_{\mathbb{T}, \mathcal{U}}(\Sigma^{-W} S^0)$$

The last term in this sequence vanishes, but the first term is isomorphic to $\mathbb{Z}$. As in the proof of 4.4, the Seiberg-Witten construction describes a homomorphism $\pi^0_{\mathbb{T}, \mathcal{U}}(\Sigma^{-W} Ci) \to \mathbb{Z}$. The choice of a "chamber" in computing the Seiberg-Witten invariant amounts to the choice of a null homotopy of the restriction $S^0 \to S^{H^+(X; \mathbb{R})}$

of the monopole map to the fixed point set. Such a null homotopy gives rise to a lift of the class of the monopole map to an element in $\pi^0_{\mathbb{T},\mathcal{U}}(\Sigma^{-W}Ci)$. The wall-crossing formulas mentioned above can be understood as describing the degree of the composite map

$$\mathbb{Z} \cong \pi^{-1}_{\mathbb{T},\mathcal{U}}(\Sigma^{-W}S^0) \to \pi^0_{\mathbb{T},\mathcal{U}}(\Sigma^{-W}Ci) \to \mathbb{Z}.$$

4.5. **Summary.** To summarize the preceeding discussion, let $\pi^0_{\mathbb{T},\mathcal{U}}(Q, Q^{\mathbb{T}})$ denote the group $\mathrm{colim}_{U \subset \Gamma(S^-)} \pi^0_{\mathbb{T},\mathcal{U}}(\Sigma^{-W}Ci)$. Note that the spectrum $\Sigma^{-W}Ci$ depends on the chosen presentation $E - \underline{U}$ for the virtual index bundle over $Pic^{\mathfrak{s}}(X)$. However, the group above by construction is independent of the chosen linear subspace $U \subset \mathcal{U}$. The groups $\pi^0_{\mathbb{T},\mathcal{U}}(Q(U))$ for big enough U become isomorphic, but not in a natural way. When writing $\pi^0_{\mathbb{T},\mathcal{U}}(Q)$, we tacidly fix some large $U \subset \Gamma(S^-)$.

Theorem 4.5. *The monopole map $\mu : \mathcal{A} \to \mathcal{C}$ for an oriented 4-manifold X with spin^c-structure $\mathfrak{s}$ defines an element in the equivariant stable cohomotopy group $\pi^0_{\mathbb{T},\mathcal{U}}(Q(X,\mathfrak{s}))$, which fits into an exact sequence*

$$\pi^{st}_1(S^{H^+(X;\mathbb{R})}) \xrightarrow{\alpha} \pi^0_{\mathbb{T},\mathcal{U}}(Q(X,\mathfrak{s}), Q(X,\mathfrak{s})^{\mathbb{T}}) \xrightarrow{\beta} \pi^0_{\mathbb{T},\mathcal{U}}(Q(X,\mathfrak{s})) \xrightarrow{\gamma} \pi^{st}_0(S^{H^+(X;\mathbb{R})}).$$

The Seiberg-Witten homomorphism $h : \pi^0_{\mathbb{T},\mathcal{U}}(Q(X,\mathfrak{s}), Q(X,\mathfrak{s})^{\mathbb{T}}) \to \mathbb{Z}$ is determined by the choice of a homology orientation and relates the monopole class to the integer valued Seiberg-Witten invariant $h\beta^{-1}([\mu])$ in case $b^+ > 1$. For $b^+ = 1$, the choice of a chamber determines a lift $[\mu]^{rel} \in \beta^{-1}([\mu])$ and $h([\mu]^{rel})$ is the corresponding Seiberg-Witten invariant. The degree of the map $h\alpha$ describes the effect of wall-crossing on the Seiberg-Witten invariant. In case $b^+ = 0$, finally, $\gamma([\mu]) = 1$.

5. Kähler, symplectic and almost complex manifolds

The current knowledge about differentiable structures on four-dimensional manifolds builds on the fact that the gauge theoretic invariants are closely related to the Cauchy-Riemann equations. Witten explained how in the case of Kähler surfaces Seiberg-Witten invariants can be determined by complex analytic methods. Taubes modified the arguments for the case of symplectic manifolds. Various mathematicians consequently studied Seiberg-Witten invariants for Kähler and symplectic manifolds. Cutting-and-pasting methods were developed to transfer these computations to other almost complex manifolds. These efforts resulted in a diverse and fascinating picture.

The refined invariants have little to add to this direction in four-manifold theory. This section intends to explain why. For the sake of brevity, let's focus on central aspects and let's assume $b^+ > 1$ in this section. As noted in the first section, a spin^c-structure is equivalent to a stably almost complex structure on the tangent bundle of a four-manifold. In particular, an almost complex manifold comes with a canonical spin^c-structure $\mathfrak{s}_{can}$. Any other spin^c-structure on the underlying oriented 4-manifold is of the form $\mathfrak{s}_{can} \otimes L$ for some $L \in H^2(X;\mathbb{Z})$, represented by a line bundle on X. With this convention the first Chern class of $\mathfrak{s}_{can}$ is minus the first Chern class K_X of the cotangent bundle.

Theorem 5.1. ([44], [40]) *Let X be a symplectic four-manifold with $b^+ > 1$. The Seiberg-Witten invariant for the canonical spin^c-structure $\mathfrak{s}_{can}$ is ± 1. Furthermore, Serre-duality holds in the following form:*

$$SW(\mathfrak{s}_{can} \otimes L) = \pm SW(\mathfrak{s}_{can} \otimes (K_X - L)).$$

Theorem 5.2. ([44], [41]) *Let X be a symplectic four-manifold with $b^+ > 1$. If for some $L \in H^2(X; \mathbb{Z})$ the Seiberg-Witten invariant of $\mathfrak{s}_{can} \otimes L$ is nonvanishing, then this spin^c-structure corresponds to an almost complex structure.*

Witten and Taubes actually prove more than is stated in these theorems: The monopole map is not surjective, unless there is a pseudo-holomorphic curve in X which is Poincaré dual to the class L. The result follows by the application of adjunction inequalities [34]. By remark (2.2), we get as an immediate consequence:

Corollary 5.3. *Let X be an oriented four-manifold with $b^+ > 1$, which admits a symplectic structure. If the stable cohomotopy invariant $[\mu] \in \pi^0_{\mathbb{T},\mathcal{U}}(Q)$ in 4.4 is non-vanishing for some spin^c-structure $\mathfrak{s}$ on X, then $\mathfrak{s}$ describes an almost complex structure on X.*

The refined invariants, when applied to symplectic manifolds, carry exactly the same information as the Seiberg-Witten invariants. This is a consequence of 5.3 and the following statement.

Proposition 5.4. *Let X be an almost complex four-manifold with $b^+ > 1$. Then the homomorphism $\pi^0_{\mathbb{T},\mathcal{U}}(Q) \to \mathbb{Z}$ in 4.4 comparing the Seiberg-Witten invariant with its refinement is an isomorphism.*

Proof. For an almost complex 4-manifold, the "virtual dimension of the moduli space" k is zero (1). The construction of the comparison homomorphism in 4.1 and 4.4 considers a map from a pair $(TE/\mathbb{T}, TE^{\mathbb{T}})$ of spaces to a sphere. The integer k is exactly the difference of the dimensions of $TE/\mathbb{T}$ and the sphere. The dimensions being equal and $(TE/\mathbb{T}, TE^{\mathbb{T}})$ being a connected and oriented manifold relative to its boundary, one can apply a classical theorem of Hopf. It states that the homotopy classes of such maps are classified by their degree. $\square$

So in order to test, whether the refined invariants are of any use, we have to leave the by now familiar world of symplectic or at least almost complex 4-manifolds and enter the jungle.

6. Some stable cohomotopy groups

The groups $\pi^0_{\mathbb{T},\mathcal{U}}(Pic^\mathfrak{s}(X); \mathrm{ind}\, l)$ seem to be at least as hard to compute as the stable homotopy groups of spheres. Let's restrict to the simplest cases. In particular, let's only consider 4-manifolds X with vanishing first Betti number and $b^+ > 1$. The groups then are then determined by the index of the linearization l of the monopole map. We will write $\pi^0_{\mathbb{T},\mathcal{U}}(\mathrm{ind}\, l)$ for short. The index of the Dirac operator is denoted by $d = \mathrm{ind}_{\mathbb{C}}(D) = \frac{c^2 - \mathrm{sign}(X)}{8}$. The virtual dimension (1) of the moduli space is $k = 2d - b^+ - 1$.

Proposition 6.1. ([7]) *Let X be a K-oriented, closed 4-manifold with vanishing first Betti number and $b^+ > 1$. The stable equivariant cohomotopy group $\pi^0_{\mathbb{T},\mathcal{U}}(\operatorname{ind} l)$ is isomorphic to the nonequivariant stable cohomotopy group $\pi^{b^+-1}(P(\mathbb{C}^d))$ of the complex $(d-1)$-dimensional projective space. This group vanishes for $k < 0$. It is isomorphic to $\mathbb{Z} \oplus A(k,d)$, if $k \geq 0$ is even, and to $A(k,d)$ otherwise. Here $A(k,d)$ denotes a finite abelian group. For any prime p, the p-primary part of $A(k,d)$ vanishes for $k < 2p - 3$. For $k \leq 4$, the groups $A(k,d)$ can be described as follows:*

- $A(0,d) \cong A(4,d) = 0$.
- $A(1,d) \cong A(2,d)$. *For even d these groups are isomorphic to $\mathbb{Z}/2$, otherwise they vanish.*
- *The 2-primary part of $A(3,d)$ is a cyclic group, depending on the congruence class of d modulo 8. The order of the group is $8, 0, 2, 4, 4, 0, 2, 2$ for the congruence classes $0, 1, 2, \ldots$.*
- *The 3-primary part of $A(3,d)$ is of order 3 if d is divisible by 3 and else vanishes.*

The proof of the first statement uses the sequence (9), which in this situation by excision is a part of the long exact cohomotopy sequence for the pair $(D(\mathbb{C}^d) \sqcup *, S(\mathbb{C}^d) \sqcup *)$ consisting of the unit ball and and its bounding sphere in the complex vector space $\mathbb{C}^d$ with an extra base point added. The $\mathbb{T}$-action on the sphere is free, so we may apply (3.5) to get the result. The Atiyah-Hirzebruch spectral sequence accounts for the rest of the statement.

Instead of chasing through technicalities, let's try to understand in an informal way, how to represent elements in these groups for small k. Recall the structure of the stable homotopy groups of spheres in low dimensions. The group $\pi^{st}_n(S^0)$ is cyclic for $n \leq 5$. It is infinite for $n = 0$, of order 2 for $n = 1$ or 2, of order 24 for $n = 3$ and zero else in this range. For $n = 1$ and 3, these groups are generated by Hopf maps $S(\mathbb{F}^2) \to P(\mathbb{F}^2)$ for $\mathbb{F} = \mathbb{C}$ and $\mathbb{F} = \mathbb{H}$, denoted by η and ν. These generators satisfy the relation $\eta^3 = 12\nu$.

First consider the map obtained by forgetting the $\mathbb{T}$-action. This homomorphism

$$f : \pi^0_{\mathbb{T},\mathcal{U}}(\operatorname{ind} l) \to \pi_{k+1}(S^0).$$

associates to a $\mathbb{T}$-equivariant map between $\mathbb{T}$-representation spheres its underlying nonequivariant map. In the case $k = 0, d = 2$, the group

$$\pi^0_{\mathbb{T},\mathcal{U}}(\operatorname{ind} l) \cong \{S^{\mathbb{C}^2}, S^3\}^{\mathbb{T}}_{\mathcal{U}} \cong \pi^2(P(\mathbb{C}^2)) \cong \mathbb{Z}$$

is generated by the unreduced suspension of the Hopf map η. For $k = 0$ and general d, the generator of $\pi_{\mathbb{T},\mathcal{U}}(Q) \cong \mathbb{Z}$ is mapped to $(d-1)\eta$.

The collapsing map $P(\mathbb{C}^d) \to P(\mathbb{C}^d)/P(\mathbb{C}^{d-1}) \cong S^{2d-2}$ induces a homomorphism

$$c : \pi_k(S^0) = \pi^{b^+-1}(S^{2d-2}) \to \pi^{b^+-1}(P(\mathbb{C}^d)) \cong \pi_{\mathbb{T},\mathcal{U}}(\operatorname{ind} l).$$

This map turns out to be an isomorphism for $k = 0$ and surjective onto the torsion subgroup $A(k,d)$ for $0 < k \leq 4$. The composite map $f \circ c : \pi_k(S^0) \to \pi_{k+1}(S^0)$ is multiplication by $(d-1)\eta$.

Finally consider the Hurewicz map

$$\pi^{b^+-1}(P(\mathbb{C}^d)) \to H^{b^+-1}(P(\mathbb{C}^d)).$$

If an element in $\pi^{b^+-1}(P(\mathbb{C}^d))$ represents a monopole map, then the image of this element under the Hurewicz map is a multiple of the generator in the cohomology group in the respective dimension. This multiplicity is the Seiberg-Witten invariant.

The Hurewicz map is neither surjective nor injective, the kernel being torsion. The non-injectivity issue makes the stable cohomotopy invariant a true refinement of Seiberg-Witten invariants. This will be addressed in the next sections. Non-surjectivity implies that, depending on k and d, the Seiberg-Witten invariants automatically satisfy certain divisibility conditions. The index of the image of the Hurewicz map $\pi^{2m}(P(\mathbb{C}^{m+n})) \to H^{2m}(P(\mathbb{C}^{m+n}))$ for $m, n \geq 0$ is known to be the stable James number $U(-m, n)$ (cf. [9], Remark 2.7). These James numbers can be defined in a more general setup and appear in various geometric situations. K-theory methods provide an estimate for them, which conjecturally is sharp:

Theorem 6.2. ([9]) *The power series in z with rational coefficients*

$$\left(\frac{z}{\log(1+z)}\right)^m,$$

when multiplied with $U(m, n)$, becomes integral modulo z^n.

7. Intermezzo

This chapter aims at sensitizing for some snags one should be aware of when working in this field. One concerns a misinterpretation of the Pontrijagin-Thom construction, another the proper use of homotopy categories.

The main difference between the familiar approach to gauge theory and the homotopy approach is the replacement of spaces by maps. The Pontrijagin-Thom construction provides a perfect and well-known duality between the concepts "stable homotopy classes of maps between spheres" and "bordism classes of framed manifolds". At first glance, this duality suggests stable maps to contain equivalent information as localized data in the form of moduli spaces together with suitably specified normal bundle data. This idea is particularly appealing to anybody working in gauge theory, since the use of localized data –often in form of characteristic classes– is a main trick of the trade. I propose to dispose of this idea as quickly as possible, since it is prone to deception and self-deception. Here is a much too long discussion; for related topics compare [1], ch. 6.

7.1. **Equivariant transversality.** One minor reason is due to the fact that the Pontrijagin-Thom correspondence fails in general in an equivariant setting due to the fact that transversality arguments don't work in sufficient generality.

7.2. **A variant on the Eilenberg-swindle.** More seriously, the information cannot possibly localize as suggested above. The reason is as puzzling as it is simple: Any two framings on a bundle by their very definition are isomorphic. Framings can be distinguished only embedded in a surrounding space.

But how to keep control over framings when changing the surrounding space? The default surrounding space we are dealing with is a Hilbert space. In order to get into business, we have to reduce to finite dimensions. And, to get this straight, the only natural way is by linear projection. Indeed, such projections are used in the

proof of 2.1. Now comes the point: Embedded framed manifolds are extremely ill behaved under projections.

Let's look at this in more detail. Embed S^1 as the $\mathbb{T}$-orbit of a nonzero element in $\mathbb{C}^d$ and fix a framing of an affine normal disk to a given point in S^1. Using the $\mathbb{T}$-action on $\mathbb{C}^d$, this framing extends to a framing of the normal bundle of S^1. By equivariant (here it is okay) Pontrijagin-Thom, this framing corresponds to a generator in the corresponding equivariant stable homotopy group, which happens to be isomorphic to $\mathbb{Z}$, as we have seen in the preceeding chapter. Now consider a generic projection $\mathbb{C}^d \to \mathbb{C}^{d-1}$. This is a $\mathbb{T}$-equivariant map and the $\mathbb{T}$-equivariant normal framing in d complex dimensions is equivariantly projected to one constructed the same way in complex $d-1$ dimensions, which also represents a generator in the corresponding group. The disastrous effect on the framing becomes apparent only after forgetting the $\mathbb{T}$-action. As explained in the preceeding chapter, nonequivariantly the constructed framing of the embedding in $\mathbb{C}^d$ is $(d-1)\eta \in \pi_1^{st}(S^0) \cong (\mathbb{Z}/2)\eta$. So it is trivial for odd d and nontrivial for even. In particular, when projecting along an infinite dimensional Hilbert space in an uncontrolled manner, we systematically do Eilenberg-swindles. There are several ways to deal with this. I'll explain some commonly used ones.

7.3. Equivariance to the rescue.
The way to gain control is by the use of the stable map representing the framing. Let's do that. This is an equivariant map $S^{\mathbb{C}^d} \to S^{2d-1}$. From the equivariant picture it is clear that this map has nothing to do with equivariant maps $S^{\mathbb{C}^{d-1}} \to S^{2d-3}$: Projection should correspond to desuspension. But considering source and target, we immediately realize that if our example were a desuspension, then it were along different $\mathbb{T}$-representations on either side of the map. The lesson should be that only by holding to the map as a double-entry book-keeping device, we can tell legal and harmless projections (desuspensions) from the illegal and harmful. But actually, in our case this is not enough.

7.4. Universes to the rescue.
Let's take a closer look at the example just discussed and let's forget that there was a $\mathbb{T}$-action. As pointed out, linear projections should correspond to desuspensions. But if we forget the $\mathbb{T}$-action, the linear projections in the example on both sides are real linear along an $\mathbb{R}^2$. As we have seen, they cannot correspond to desuspensions. Intuitively, the problem is easy to understand: In the source, we are trying to desuspend a "moving frame", whereas in the target, we want to desuspend a "fixed frame". Now that we have excluded representation theory to act as a savior, we need a replacement to convey that idea. The notion of a universe, which seems to go back to Peter May, is such a replacement. The point here is that the projection above along $\mathbb{R}^2$ does not factor through a projection along $\mathbb{R}^1$ as it should. If one uses universes, this feature is built in.

7.5. On the usage of spectra I.
I want to present a way how not to define the refined invariants: This uses the spectrum of a self-adjoint elliptic operator, acting on a Hilbert space $\mathcal{U}$. After choosing an oriented basis for eigenspaces, we get a canonical embedding $\mathbb{R}^\infty \to \mathcal{U}$, which we may use to make suspensions ordered by

the integers instead of finite dimensional linear subspaces of $\mathcal{U}$. This is okay if one does not change the operator.

The snag appears if one wants to change the operator. Let's do that, say by changing a metric used to define it. At first sight this looks controllable: A small change of the operator will result in a small change of the eigenvalues, so locally, up to "canonical" homotopy, this should define a "canonical" homotopy equivalence between the sphere spectra indexed by the integers.

Will this stand up to scrutiny? Assume we have a closed path of operators such that the eigenspaces for eigenvalues in a fixed interval constitute a bundle over $B = S^1$. It may happen that the bundle for a chosen set of eigenvalues is not orientable. Following the "canonical" homotopy equivalences of the sphere along the circle, we obtain that the identity map over the base point is "canonically" homotopic to minus the identity map. This is not what we want.

But, the space of metrics is contractible. So we may always extend the operator to an operator parametrized by a disk. In the critical cases this will involve other eigenspaces than the ones we started with. So only very special arrangements of eigenspaces will be "admissible" for the argument. And which arrangements are "admissible" may depend heavily on the chosen extension of the operator on the disk. There may exist no "admissible" arrangement that works in all situations.

Orientation is governed by a determinant line bundle, which exists in the Fredholm setting. So, indeed, there may be a way to coherently enforce all such bundles over B to be orientable. I don't know any, but let's suppose we found one. Then, as bundles over B, they are trivial. However, there are two trivializations up to homotopy to choose from. If we pick the wrong one, we will have the following phenomenon: Using the trivialization, we may parallel transport an embedded S^1 with framed normal bundle in the fiber over a point in B once around the loop B. This parallel transport changes the framing.

But, the space of metrics is contractible. Indeed, if the operator is such that the bundle over B extends to a bundle of eigenspaces over the disc, then this would pick a trivialization. However, there may be a different extension of the operator to the disk such that we get a trivialization only if we add a 2-dimensional eigenspace. The two trivializations obtained that way need not be the same, as the example (7.2) shows. Which to choose?

Let's stop here. Who ever desires to use eigenspaces of self-adjoint operators to define homotopy objects is kindly asked not to rely on bluff and belief, but on reason. I cannot see, how to create well-defined mathematical objects this way and I doubt it is possible. I recommend universes instead.

7.6. **Why do universes work?** The discussion above shows well-definedness of the refined invariants to be a non-trivial issue. The following argument is not based on the contractibility of some parameter space, but on the contractibility of the orthogonal group of Hilbert space. If we take a path in our parameter space (metrics, $spin^c$-connections), then we will get a bundle of universes over that path. The theorem of Kuiper [29] shows that this is a trivial bundle and has a unique trivialization up to homotopy. A trivialization identifies the universes defined for different parameters. Such an identification of universes provides for a change-of-universe isomorphism of the stable cohomotopy groups defined with respect to

the respective universes. A trivialization homotopic modulo end points to the one chosen will induce the same isomorphism of stable cohomotopy groups. This uses 2.1 for the parameter space $B \times [0,1] \times [0,1]$. In particular, by taking closed paths, we get that the invariants are well defined.

7.7. **On the usage of spectra II.** Finally, I want to point out a reliable avenue to create nonsense. Is it possible to construct (homotopy types of) spectra out of spaces, which themselves are only defined up to homotopy? That means, all spaces are defined up to homotopy, the suspensions are defined up to homotopy, the compatibility condition (5) holds only up to homotopy. The answer in general is: No. This would amount to a lift from the homotopy category of topological spaces to the category of topological spaces. This problem has been addressed in work of Dwyer and Kan, compare e.g. [15], [16]. To see the problems, just assume for each $U \subset \mathcal{U}$, the space A_U to be a sphere homotopy equivalent to S^U. When trying to prove well-definedness of the identity map, not only similar problems as above turn up, but also higher dimensional phenomena. There is no magic to cure this problem.

Since I am using [10] as a reference, I should point out that his definition of spectra looks similar to the one I am criticizing. It actually is different: The author wisely only uses complex representations as suspension coordinates. Because of the implicit $\mathbb{T}$-equivariance (7.3) this gets rid of all the complications I lamented about. Moreover, the author is only interested in spectra as realizing equivariant homology and cohomology functors on spaces. He does not define a category of spectra and in particular he does not define maps of spectra, thus avoiding any discussion about the indicated higher dimensional phenomena.

Not all authors have taken this problem in homotopy theory serious. Sadly enough, it renders a considerable part of the literature in this subject useless.

8. Gluing results

8.1. **Gluing along positive curvature.** Connected sums of oriented 4-manifolds have vanishing Seiberg-Witten invariants, unless one of the summands has negative definite intersection form. The same statement holds for Donaldson invariants. This fits very well with known stability results on simply connected 4-manifolds: A theorem of Wall [43] states that if any two differentiable 4-manifolds are homotopy equivalent, then after taking connected sum with sufficiently many copies of $S^2 \times S^2$, the resulting manifolds will be diffeomorphic. In many cases it is known that already one copy is "sufficiently many". For example, complete intersections or elliptic surfaces are almost completely decomposable [31]. That means, the result of taking connected sum with a single complex projective plane is diffeomorphic to a connected sum of projective planes, taken with both standard and reversed orientation. Simon Donaldson defined in [13] $mod\, 2$-polynomial invariants, which potentially could distinguish different structures on connected sums. However, no examples were found.

The stable cohomotopy invariants don't vanish in general for connected sums. This shows that they are true refinements of Seiberg-Witten invariants. The connected

sum theorem [6] states that for a connected sum $X_0 \# X_1$ of 4-manifolds, the stable equivariant cohomotopy invariant is the smash product of the invariants of its summands. It is straightforward to compute explicit examples.

A precise statement of the theorem constitutes already a major part of its proof. We will discuss a slightly more general setup. Let X be the disjoint union of a finite number, say n, of closed connected Riemannian 4-manifolds X_i, each equipped with a K-theory orientation. Suppose each component contains a separating neck $N_i \cong Y \times [-L, L]$. So it is a union

$$X_i = X_i^- \cup X_i^+$$

of closed submanifolds with common boundary $\partial X_i^\pm = Y \times \{0\}$. Here Y denotes a 3-manifold with a fixed Riemannian structure. The length $2L > 2$ of the neck is considered a variable. For an even permutation τ of the indices, let X^τ be the manifold obtained from X by interchanging the positive parts of its components, that is

$$X_i^\tau = X_i^- \cup X_{\tau(i)}^+.$$

Next comes the question of whether and how K-orientations glue. In order to be able to glue, we of course need the following

Assumption: *The K-orientations on all components X_i, when pulled back along the inclusion $Y \times [-L, L] \xrightarrow{\sim} N_i \hookrightarrow X_i$, lead to the same K-orientation.*

This assumption is automatically satisfied in case Y is an integral homology sphere. In general, in order to get a well-defined K-orientation on the manifold X^τ, it does not suffice to fix an isomorphism class, but we also have to fix identifications. Note that the gauge group $map(Y \times [-L, L], \mathbb{T})$ acts freely on the set of all such identifications. If the gauge group is connected, any such identification will give the same K-orientation on X^τ. We can enforce connectedness by the

Assumption: *Let Y have vanishing first Betti number.*

It turns out that we will have to put much stronger assumptions on the geometry of Y in order to prove the gluing theorem. So we need not discuss this tricky issue at this point. Under these assumptions, a K-theory orientation of X uniquely induces by gluing one on X^τ. A main ingredient for the gluing setup is a change of universe isomorphism $V_Y : \mathcal{U} \to \mathcal{U}^\tau$. Its explicit construction uses a smooth path

$$\psi : [-1, 1] \to SO(n)$$

starting from the unit, i. e. $\psi(-1) = \mathrm{id}$, and ending at τ, considered as the permutation matrix $(\delta_{i, \tau(j)})_{i,j} \in SO(n)$. Suppose we are given a bundle over X such that the restrictions over the necks are identified with a bundle F over $Y \times [-L, L]$. Using these identifications, the restrictions of the bundle to $X_i^\pm$ glue together to a bundle over X^τ. Sections of the given bundle, when restricted over the neck, can be viewed as a section of the bundle $\oplus_{i=1}^n F$ over $Y \times [-L, L]$. Consider the path ψ as rotation of the components of this bundle. Rotating via ψ a given section of a bundle over X results in a section of the glued bundle over X^τ. This gluing construction, applied to forms and spinors on X, defines fiberwise linear bundle isomorphisms $V_Y : \mathcal{A} \to \mathcal{A}^\tau$ and $V_Y : \mathcal{C} \to \mathcal{C}^\tau$ of the Hilbert space bundles over a suitably defined identification $Pic^{\mathfrak{s}}(X) \xrightarrow{\cong} Pic^{\mathfrak{s}}(X^\tau)$. The following theorem is formulated in [6] only for the cohomotopy groups in 4.1 and the case $Y = S^3$. The

proof extends without further changes to the version in 4.4 and to positively curved manifolds Y, i.e. quotients of the sphere.

Theorem 8.1. *Let Y be a manifold with positive Ricci and in particular scalar curvature. Then the change of universe isomorphism*

$$V_Y : \pi^0_{\mathbb{T},\mathcal{U}}(Pic^{\mathfrak{s}}(X); \operatorname{ind} l) \to \pi^0_{\mathbb{T},\mathcal{U}^\tau}(Pic^{\mathfrak{s}}(X^\tau); \operatorname{ind} l^\tau)$$

identifies the monopole classes of X and X^τ for corresponding K-theory orientations.

The theorem claims the diagram

$$
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{\mu} & \mathcal{C} \\
V \downarrow & & \downarrow V \\
\mathcal{A}^\tau & \xrightarrow{\mu^\tau} & \mathcal{C}^\tau
\end{array}
$$

to commute up to homotopy, i.e. there is a path in $\mathcal{P}_l(\mathcal{A}, \mathcal{C})$ connecting the maps μ and $V^{-1}\mu^\tau V$. The difference between the two maps is a compact operator. So the homotopy need only change the compact summand in the monopole map such that at any time during the homotopy the boundedness condition (1.3) remains satisfied. Control is achieved by the use of Weitzenböck formulas for both the Dirac operator and the covariant derivative. Positivity of scalar and Ricci curvature, respectively, along the neck provide the necessary estimates on the spinor and form components during the homotopy. The estimates on spinor and forms finally are tuned by neckstretching. So the theorem holds for sufficiently large L and hence for any $L > 1$. The proof in [6] actually constructs a path in a slightly bigger space than $\mathcal{P}_l(\mathcal{A}, \mathcal{C})$. This can be avoided by the use of the homotopy (6).

To apply this theorem, let's spell out the following elementary observation.

Proposition 8.2. ([6]) *Let X be the disjoint union of a finite number of K-oriented 4-manifolds X_i. Then the Thom spectrum $T(\operatorname{ind} l)$ of the index bundle over $Pic^{\mathfrak{s}}(X)$ is the smash product of the corresponding spectra $T(\operatorname{ind} l_i)$ associated to the components and the stable cohomotopy class of the monopole map of X is the smash product*

$$[\mu(X, \mathfrak{s})] = \wedge_{i=1}^n [\mu(X_i, \mathfrak{s}_i) \in \pi^0_{\mathbb{T}^n, \oplus \mathcal{U}_i}(Pic^{\mathfrak{s}}(X); \operatorname{ind} l)$$

of the stable cohomotopy classes associated to the respective components. The action of the torus $\mathbb{T}^n$ on the sum $\oplus_{i=1}^n \mathcal{U}_i$ is factorwise.

Note that the $\mathbb{T}$-action in 8.1 on these spectra is the diagonal one.

The proof of the connected sum formula follows from applying this theorem to the case $Y = S^3$ when X is the disjoint union of a connected sum $X_0 \# X_1$ and two copies of the 4-sphere. The manifold X^τ then will be the disjoint union of X_0, X_1 and one further copy of the 4-sphere. Using (4.5) and (3.1) it is immediate to recognize $\mu(S^4)$ as homotopic to the identity map on the sphere spectrum. In particular, we may identify the monopole class $[\mu(X_0 \# X_1, \mathfrak{s}_0 \# \mathfrak{s}_1)]$, with the monopole class

$$[\mu(X_0 \# X_1, \mathfrak{s}_0 \# \mathfrak{s}_1) \wedge \mu(S^4) \wedge \mu(S^4)] = [\mu(X_0 \# X_1, \mathfrak{s}_0 \# \mathfrak{s}_1) \wedge \operatorname{id}_{S^0} \wedge \operatorname{id}_{S^0}],$$

via some obvious change-of-universe identifications. So as a corollary to 8.1 we obtain the connected sum theorem.

Theorem 8.3. ([6]) *The gluing map V_{S^3} identifies the class $[\mu(X_0 \# X_1, \mathfrak{s}_0 \# \mathfrak{s}_1)]$ of the monopole map of the connected sum of two K-oriented 4-manifolds with the smash product $[\mu(X_0, \mathfrak{s}_0)] \wedge [\mu(X_1, \mathfrak{s}_1)]$ of the monopole classes of the summands.*

The gluing theorem applies to a further construction, which is discussed in [25], p. 411: Suppose, the K-oriented 4-manifolds X_0 and X_1 both contain a -2-curve, i.e. a smoothly embedded 2-sphere with self-intersection number -2. Cutting out tubular neighbourhoods of these -2-curves, we obtain manifolds with real projective 3-space as boundary. Using an orientation reversing diffeomorphism of the boundaries, we may glue the manifolds along their boundaries. Let $X_0 \#_2 X_1$ denote the resulting manifold.

The orientation reversing diffeomorphism permutes the two spin-structures on the real projective 3-space $P(\mathbb{R}^4)$. One of these two spin-structures extends as a spin-structure to the tubular neighbourhood of a -2-curve. This property distinguishes the two spin-structures. The other spin structure extends to a $spin^c$-structure on the tubular neighbourhood, the determinant line bundle of which has degree congruent 2 mod 4, when restricted to the -2-curve.

Gluing two copies of tubular neighbourhoods of -2-curves by the use of an orientation reversing diffeomorphism of the boundaries, results in a manifold N. This manifold can also be recognized as the manifold $N = \overline{P(\mathbb{C}^3)} \# P(\mathbb{C}^3)$ obtained by reversing the orientation on the connected sum of two copies of the complex projective plane. There are four $spin^c$-structures on N for which the monopole map is homotopic to the identity map on the sphere spectrum. This again is immediate from (4.5) and (3.1). Exactly the same argument as in 8.3, with S^4 replaced by N, thus proves:

Theorem 8.4. *Let $X_0 \#_2 X_1$ be the sum of two 4-manifolds along -2-curves with $spin^c$-structure $\mathfrak{s}$. Then there are $spin^c$-structures $\mathfrak{s}_0$ and $\mathfrak{s}_1$ on X_0 and X_1, respectively, such that one of the associated first Chern classes evaluates at the corresponding -2-curve with 2, the other with 0 and $\mathfrak{s} = \mathfrak{s}_0 \#_2 \mathfrak{s}_1$. The gluing map $V_{P(\mathbb{R}^4)}$ identifies the class of the monopole map $[\mu(X_0 \#_2 X_1, \mathfrak{s})]$ with $[\mu(X_0, \mathfrak{s}_0)] \wedge [\mu(X_1, \mathfrak{s}_1)]$.*

Obviously, the range of applications of 8.1 is rather limited. It would be desirable to extend the stable cohomotopy approach in a well-defined manner (cf. 7.7) to manifolds with boundary explaining the behaviour under cutting and pasting.

8.2. **Applications to 4-manifolds.** The computations of the stable cohomotopy groups in (6.1) can now be combined with known results on Seiberg-Witten invariants (5.4). Most of the following statements are immediate.

Theorem 8.5. (*Vanishing results for connected sums,* [6].) *Let X be a connected sum of oriented 4-manifolds. Then the refined invariants vanish for any $spin^c$-strucure on X in the following cases:*

(1) *The refined invariants vanish for any $spin^c$-structure on one of the summands.*

(2) *There are two or more summands which are symplectic and have vanishing first Betti numbers. Furthermore, one symplectic summand X_0 satisfies $b^+(X_0) \equiv 1 \, mod \, 4$.*

(3) The manifold X has vanishing first Betti number, $b^+(X) \not\equiv 4 \mod 8$ and is a connected sum of 4 symplectic manifolds.

(4) The manifold X has vanishing first Betti number and is a connected sum of 5 symplectic manifolds.

The theorem remains true, if one replaces "symplectic" by the weaker assumption "all $spin^c$-structures with non-trivial refined invariants are almost complex".

Theorem 8.6. (*Vanishing results for sums along -2-spheres.*) *Let X_0 and X_1 be oriented 4-manifolds containing -2-spheres C_0 and C_1, respectively. Then the refined invariants vanish for any $spin^c$-strucure on the sum $X_0 \#_2 X_1$ along these spheres in the following cases:*

(1) *The refined invariants vanish for any $spin^c$-structure on X_0.*

(2) *The first Chern class of any $spin^c$-structure on X_i, for which the refined invariant is nonvanishing, gives the same number modulo 4 when evaluated on C_i (for both $i = 0, 1$).*

(3) *The first Chern classes of those $spin^c$-structures on X_i, which have non-vanishing refined invariants, span a linear subspace of $H^2(X_i; \mathbb{Q})$ on which the cup-product is positive semi-definite.*

(4) *Both X_0 and X_1 can be equipped with the structure of a minimal Kähler surface with $b^+(X_i) > 1$.*

Proof. The first two statements are immediate from 8.4: The assumptions imply one of the two factors in the smash product to vanish. The fourth statement is a special case of the third. The proof of 3 uses the following, well-known fact: Complex conjugation in a small tubular neighbourhood of a -2-sphere extends to an automorphism of the 4-manifold X_i which is constant outside a larger tubular neighbourhood. The effect in second cohomology is a reflection on the hyperplane perpendicular to the Poincaré dual $PD(C_i)$ of the -2-curve. If there was a $spin^c$-structure with non-vanishing refined invariant, whose first Chern class is not perpendicular to $PD(C_i)$, then $PD(C_i)$ were a linear combination of the first Chern classes of this $spin^c$-structure and its reflected $spin^c$-structure. This would contradict the assumption. As a consequence, the second condition is satisfied, the number modulo 4 being 0. $\square$

Question 8.7. *Is there a minimal symplectic 4-manifold with $b^+ > 1$ for which the first Chern classes of $spin^c$-structures with non-vanishing Seiberg-Witten invariants span a linear subspace of $H^2(X; \mathbb{Q})$ which is not positive semidefinite?*

Here are some general non-vanishing results. Of course, no manifold can be on both a vanishing list as above and a non-vanishing list. This has some non-trivial implications. Note that the assumptions are met by symplectic manifolds.

Theorem 8.8. (*Non-vanishing results for connected sums, [6].*) *There is a $spin^c$-structure on the oriented 4-manifold X for which the associated refined invariant is non-vanishing, if one of the following holds:*

(1) *The manifold X is a connected sum $X = X_0 \# X_1$ of a manifold X_0, which admits a $spin^c$-structure with non-vanishing refined invariant, and a manifold X_1 with $b^+(X_1) = 0$.*

(2) *The manifold X has vanishing first Betti number and is a connected sum with two or three summands. For every summand X_i there is an almost complex structure for which the integer Seiberg-Witten invariant is odd and $b^+(X_i) \equiv 3 \bmod 4$.*

(3) *The manifold X is a connected sum with four summands, has vanishing first Betti number and $b^+(X) \equiv 4 \bmod 8$. For every summand X_i there is an almost complex structure for which the integer Seiberg-Witten invariant is odd and $b^+(X_i) \equiv 3 \bmod 4$.*

Proof. Only the first statement is not discussed in [6]. Using Donaldson's theorem 4.3 we can find a $spin^c$-structure on X_1 such that the virtual index bundle of the Dirac operator over $Pic^{\mathfrak{s}}(X_1)$ has rank 0. The inclusion of a point in $Pic^{\mathfrak{s}}(X_1)$ induces a restriction map $\pi^0_{\mathbb{T},\mathcal{U}}(Pic^{\mathfrak{s}}(X_1); \operatorname{ind} l) \to \pi^0_{\mathbb{T},\mathcal{U}}(S^0)$. The image of the monopole class is the identity map. $\square$

The information retained in the refined invariants of connected sums is much more detailed than these sweeping vanishing and non-vanishing theorems might suggest. To get an impression, let's consider connected sums of certain elliptic surfaces which had been classified [5] up to diffeomorphism with methods from Donaldson theory. Note that in each of the two homeomorphism classes of such elliptic surfaces there are infinitely many diffeomorphism classes.

Corollary 8.9. ([6]) *Suppose the connected sum $\#_{i=1}^4 E_i$ of simply connected minimal elliptic surfaces of geometric genus one is diffeomorphic to a connected sum $\#_{j=1}^n F_j$ of elliptic surfaces. Then $n = 4$ and the F_j and the E_i are diffeomorphic up to permutation.*

Ishida and LeBrun [27], [28] pointed out some differential geometric applications of the connected sum theorem. In particular they proved non-existence statements for Einstein metrics on connected sums of algebraic surfaces.

9. ADDITIONAL SYMMETRIES

9.1. **Spin structures.** The case of spin structures was pioneered by Furuta [20]. The key observation is that for a spin 4-manifold X the monopole map is actually $Pin(2)$-equivariant, where $Pin(2) \subset Sp(1) \subset \mathbb{H}$ is the normalizer of the maximal torus $\mathbb{T} \subset \mathbb{C} \subset \mathbb{H}$ in $Sp(1)$. This subgroup is generated by $\mathbb{T}$ and an additional element $j \in \mathbb{H}$ satisfying $j^2 = -1$ and $ij + ji = 0$.

The group $Spin(4)$ is isomorphic to the product of two copies of $Sp(1) \cong SU(2)$ and embeds as a subgroup in $Spin^c(4)$. So the $Spin^c(4)$-representations used in the definition of the monopole map naturally restrict to $Spin(4)$-representations. Considered this way as $Spin(4)$-representations, Δ^+ and Δ^- admit quaternionic structures. The Dirac operator, therefore, is $\mathbb{H}$-linear.

This additional structure is not preserved by the monopole map: Consider the induced action of $Sp(1)$ on the space of all $Spin(4)$-equivariant quadratic maps $\Delta^+ \to \Lambda^+$. The isotropy group of the term σ in the definition of the monopole map is $\mathbb{T}$. The normalizer of the torus interchanges σ and $-\sigma$. This indicates, for which action and which group the monopole map can be made equivariant.

Taking the *spin*-connection A as the background $spin^c$-connection, we can define a $Pin(2)$-action on the spaces $\mathcal{A}$ and $\mathcal{C}$ used in the definition of the monopole map:

The group acts via the quaternionic structure on the sections of the quaternionic bundles S^+ and S^-. The element j acts via multiplication by -1 on both forms and $spin^c$-connections (after identifying the space of connections with $A + i\Omega^1(X)$. The monopole map (3) indeed is equivariant with respect to this $Pin(2)$-action.

In this setup, our standard universe $\mathcal{U} = \Gamma(S^-) \oplus \Omega^0(X) \oplus H^1(X;\mathbb{R}) \oplus \Omega^+(X)$ will not contain trivial $Pin(2)$-representations. As a consequence, equivariant cohomotopy $\pi^0_{Pin(2),\mathcal{U}}(Pic^{\mathfrak{s}}(X); \operatorname{ind} l)$ in general does not carry a group structure; it is just a set which even may be empty. Indeed, the main result in [20] in effect proves emptiness of this set in certain cases.

In order to get groups, we may simply enlarge the universe $\mathcal{V} = \mathcal{U} \oplus H$ by adding an infinite dimensional Hilbert space H with trivial $Pin(2)$-action. The change-of-universe map $\pi^0_{Pin(2),\mathcal{U}}(Pin^{\mathfrak{s}}(X); \operatorname{ind} l) \to \pi^0_{Pin(2),\mathcal{V}}(Pin^{\mathfrak{s}}(X); \operatorname{ind} l)$ can be viewed as induced by smash product with the identity element in $\pi^0_{Pin(2),H}(S^0)$ and turns out [36] to be injective in case $b_1 = 0$; the image is characterized in algebraic terms. In particular, there is no loss of information by this change-of-universe, but a gain of convenient algebraic structure.

Theorem 9.1. ([20]) *Let X be a spin 4-manifold with $sign(X) < 0$. Then the second Betti number of X satisfies the inequality*

$$b_2(X) \geq 2 - \frac{10}{8} \, sign(X).$$

Proof. The inclusion $A \hookrightarrow Pic^0(X)$ of the *spin*-connection induces a restriction map $\pi^0_{Pin(2),\mathcal{V}}(Pic^{\mathfrak{s}}(X); \operatorname{ind} l) \to \pi^0_{Pin(2),\mathcal{V}}(\operatorname{ind} l)$. The index of l is $\frac{-sign(X)}{16}\mathbb{H} - H^+(X;\mathbb{R})$. In order to apply the K-theory degree formula, we need to complexify these $Pin(2)$-representations. So, consider the square of the monopole map

$$\nu = [\mu(X)] \wedge [i\mu(X)] \in \pi^0_{Pin(2),\mathcal{V}+i\mathcal{V}}\left(\Sigma^{-H^+(X;\mathbb{R})\otimes\mathbb{C}}(S^{\mathbb{H}^d})\right).$$

The element $j \in Pin(2)$ acts by multiplication with -1 on the $Pin(2)$-representation $H^+(X;\mathbb{R}) \otimes \mathbb{C}$. We would like to compute the $K_{Pin(2)}$-mapping degree $a_{Pin(2)}(\nu)$ of ν via the formula (8)

$$e_{K_{Pin(2)}}(H^+(X;\mathbb{R}) \otimes \mathbb{C}) \cdot d_{Pin(2)} = a_{Pin(2)}(\nu) \cdot e_{K_{Pin(2)}}(\mathbb{H}^d),$$

which takes place in the representation ring $R(Pin(2)) \cong \mathbb{Z}[\lambda, h]/(\lambda^2 - 1, \lambda h - h)$. Here λ stands for the one-dimensional representation on which j acts by multiplication with -1 and h stands for the quaternions, viewed as a $Pin(2)$-representation. The singular cohomology mapping degree $d_{Pin(2)}$ can be computed by considering for each element in $Pin(2)$ the cohomology degree on the fixed point spheres of that element. By dimension reasons, this vanishes except for the conjugates of j. It is 1 for j itself, as by construction ν is the identity on the fixed point set. So we get $d_{Pin(2)} = \frac{1}{2}(1 - \lambda)$. The $K_{Pin(2)}$-Euler classes are computed via (7) to be $e_{K_{Pin(2)}}(H^+(X;\mathbb{R}) \otimes \mathbb{C}) = (1 - \lambda)^{b^+}$ and $e_{K_{Pin(2)}}(\mathbb{H}^d) = (2 - h)^d$. The mapping degree formula thus reads

$$\frac{1}{2}(1 - \lambda)^{b^+ + 1} = a_{Pin(2)}(2 - h)^d.$$

In the representation ring, this equality can only be satisfied, if $a_{Pin(2)}$ is of the form $a(1 - \lambda)$ for some integer a (the character on the left hand side is zero on $\mathbb{T}$!).

So we are left with the equation

$$2^{b^+ - 1}(1 - \lambda) = a2^d(1 - \lambda),$$

which can be satisfied only for $d \leq b^+ - 1$, or equivalently, $b^+ \geq 1 - \frac{sign(X)}{8}$.   □
This theorem can be sharpened a little:

Theorem 9.2. ([35],[36]) *Let X be a spin 4-manifold with $sign(X) < 0$. Then the second Betti number of X satisfies the inequality*

$$b_2(X) \geq 2a - \frac{10}{8}\, sign(X),$$

with $a = 2$, if $sign(X) \equiv 32 \bmod 64$ and $a = 3$, if $|sign(X)| \equiv 48 \bmod 64$ and $a = 1$ else. Moreover, in the case $sign(X) = -64$, one has $b_2(X) \geq 88$.

The two references correspond to two different proofs. The first one relies on results of S. Stolz and M. Crabb in $\mathbb{Z}/4$-equivariant stable homotopy. The second one imitates in principle the proof above, using more refined $KO_{Pin(2)}$-mapping degrees instead. Furuta in [21] announced that by the same methods one can prove $a = 3$ if $sign(X) \equiv 0 \bmod 64$.

The following example shows that these methods cannot be carried through to prove the so-called $\frac{11}{8}$-conjecture stating $b_2(X) \geq -\frac{11}{8}sign(X)$.

Theorem 9.3. ([36]) *There is an element in $\{S^{\mathbb{H}^5}, S^{V^{12}}\}_{\mathcal{U}}^{Pin(2)}$, where V is the real 1-dimensional non-trivial $Pin(2)$-representation.*

So the lowest rank of a potential counterexample to the $\frac{11}{8}$-conjecture, at least according to current knowledge, is $b_2 = 104$.

The connected sum theorem also works in the $Pin(2)$-equivariant setting for *spin*-manifolds. Taking connected sum of a *spin*-manifold X with $S^2 \times S^2$ amounts for the $Pin(2)$-equivariant monopole classes to multiplication with the stable cohomotopy Euler class $e(V) : S^0 \hookrightarrow S^V$ of the $Pin(2)$-representation V.

The long exact sequence for the pair of spaces $(D(V) \sqcup *, S(V) \sqcup *)$, together with the adjunction 3.3 leads for a $Pin(2)$-spectrum A to a long exact Gysin sequence

$$\ldots \to \pi_{\mathbb{T}, \mathcal{V}}^{-1}(S^V \wedge A) \to \pi_{Pin(2), \mathcal{V}}^0(S^V \wedge A) \to \pi_{Pin(2), \mathcal{V}+V}^0(A) \to \pi_{\mathbb{T}, \mathcal{V}+V}^0(A) \to \ldots.$$

The map in the middle is multiplication with the Euler class $e(V)$. The next map restricts the group action. Application to the Thom spectrum $T(\mathrm{ind}\, l)$ of the index bundle over $Pic^5(X)$ gives as an immediate consequence:

Theorem 9.4. ([36]) *Suppose the $Pin(2)$-equivariant monopole class of a spin 4-manifold X with $sign(X) < 0$ is not divisible by the Euler class $e(V)$. Then the refined Seiberg-Witten invariant of X is nonzero. In particular, if X has second Betti number $b_2(X) = -\frac{11}{8}sign(X)$ for $|sign(X)| \leq 64$ (or $b_2(X) = 104$ and $sign(X) = -80$), then X has non-vanishing refined invariants.*

The special cases of vanishing first Betti numbers and $sign(X) = -16$, $sign(X) = -32$ and $sign(X) = -48$ were obtained in [33], [22] and [23], respectively.

9.2. **Symplectic structures with** $c_1 = 0$. Let X be a K-oriented 4-manifold, which is both symplectic and spin. This means the canonical $spin^c$-structure coming with the symplectic structure has vanishing integral first Chern class. For such a manifold one can combine the considerations above with Taubes' result (5.1).

According to the Kodaira classification of complex surfaces [4] there are only three families of complex surfaces with $c_1 = 0$. The obvious families are the simply connected $K3$-surfaces and the tori. Furthermore, there are primary Kodaira surfaces with first Betti number 3. The first two families are Kähler, hence symplectic by default. Other symplectic, non-Kähler and even non-complex manifolds with $c_1 = 0$ and Betti numbers 2 and 3, were constructed [42], [18], [24].

Theorem 9.5. ([33]) *Let X be a symplectic 4-manifold with vanishing first Betti number and with trivial canonical line bundle. Then $sign(X) = -16$.*

Proof. The vanishing of c_1 forces X to be spin with $sign(X) = -16n$ and $b^+ = 4n-1$. The monopole map is $Pin(2)$-equivariant, and after moding out the $\mathbb{T}$-action as in 6.1, we obtain a stable $\mathbb{Z}/2$-equivariant map in $\{P(\mathbb{C}^{2n}), S(V^{4n-1})\}^{\mathbb{Z}/2}$. The action is free on both spaces and the spaces are of the same dimension. This allows to apply a $\mathbb{Z}/2$-equivariant version of the Hopf theorem [10], p 126. According to this theorem, the (nonequivariant) degree of such a map is determined modulo 2. As we will see, this degree, which is the Seiberg-Witten invariant, can be an odd number only in the case $sign(X) = -16$. Taubes' theorem 5.1 then completes the argument. To show that the degree is even for $n > 1$, it suffices because of Hopf's theorem to exhibit an element in $\{P(\mathbb{C}^{2n}), S(V^{4n-1})\}^{\mathbb{Z}/2}$ which has even degree. Here it is: The n-th power η^n of the $Pin(2)$-equivariant Hopf map induces a $\mathbb{Z}/2$-equivariant map $P(\mathbb{C}^{2n}) \to S(V^{3n})$. Composed with the inclusion $S(V^{3n}) \to S(V^{4n-1})$ we get a map of degree zero for $n > 1$. $\qquad\square$

Here is an immediate corollary:

Corollary 9.6. *A symplectic 4-manifold with finite fundamental group and with trivial canonical line bundle is homeomorphic to a $K3$-surface.*

It is well-known that there are infinitely many different smooth structures on the topological 4-manifold underlying a $K3$-surface. The infinitely many smooth structures which come from complex Kähler structures were classified in [5]. There are infinitely many more smooth structures which come from symplectic ones, compare [25], p. 396f. And there are again infinitely many more smooth structures which don't allow for a symplectic structure at all, compare [19]. Nevertheless, amongst all these smooth structures only the $K3$-surface seems to be known to carry a symplectic structure with $c_1 = 0$. The analogy to the Kodaira-dimension zero case in the Kodaira-classification therefore is tantalizing:

Question 9.7. *Are symplectic 4-manifolds with $c_1 = 0$ necessarily either parallelizable or $K3$-surfaces?*

9.3. **Group actions.** The stable cohomotopy approach does not rely on transversality results and therefore seems suitable for considering group actions on 4-manifolds. In the discussion below, which closely follows [39], the first Betti number of the manifolds will always be zero. A compact Lie group acting on a 4-manifold X is supposed to preserve its K-orientation.

Theorem 9.8. ([39]) *Let G act on the K-oriented 4-manifold X. There is a central extension $\mathbb{G}$ of the group G by the torus $\mathbb{T}$, such that the monopole map $\mu : \mathcal{A} \to \mathcal{C}$ is $\mathbb{G}$-equivariant. The associated element $[\mu]_G \in \pi^0_{\mathbb{G},\mathcal{U}}(\operatorname{ind} l)$ restricts to the $\mathbb{T}$-equivariant stable cohomotopy invariant.*

When considering free actions of finite groups, one gets into a situation which very much resembles Galois theory. The quotients X/H by the various subgroups $H < G$ are 4-manifolds carrying a residual action of the Weyl group $WH = N_G H/H$.

Theorem 9.9. ([39]) *Let X be a K-oriented 4-manifold with a free action of a finite group G and $H < G$ a subgroup. The set $J(H, \mathfrak{s}_X)$ of spin^c-structures on the quotient X/H which pull back to the given spin^c-structure $\mathfrak{s}_X$ on X can be canonically identified with the set of subgroups of $\mathbb{G}$ which map isomorphically to H under the projection to G. For $j \in J(H, \mathfrak{s}_X)$ the invariant $[\mu(X/H, \mathfrak{s}_j)]_{WH}$ can be identified with the restriction of $[\mu(X, \mathfrak{s})]_G$ to the fixed points of $H(j) < \mathbb{G}$.*

In particular, stable cohomotopy invariants of oriented 4-manifolds with finite fundamental group are determined by equivariant stable cohomotopy invariants of simply connected 4-manifolds.

One can combine all the restrictions to fixed points into a comparison map

$$\pi^0_{\mathbb{G},\mathcal{U}}(\operatorname{ind} l(X, \mathfrak{s})) \to \bigoplus_{(H) \leq G} \bigoplus_{J(H, \mathfrak{s}_X)} H^0(WH; \pi^0_{\mathbb{T}, \mathcal{U}^{H(j)}}(\operatorname{ind} l(X/H; \mathfrak{s}_j))).$$

Under certain conditions a general splitting result in equivariant homotopy theory implies that this comparison map is an isomorphism after localisation away from the order of the group. This splitting theorem can be applied for example if both $b^+(X/H) > 1$ holds for any subgroup of G and the index of the Dirac operator can be represented by an actual representation. So in this case kernel and cokernel are torsion groups with nonzero p-primary parts only for those primes which do divide the order of G.

Finally, let's restrict to the case of a group of prime order p. Again the case where the K-orientation on X comes from an almost complex structure is easy to handle.

Theorem 9.10. ([39]) *If the group G of prime order p acts freely on the almost complex manifold X, then the invariant $[\mu(X, \mathfrak{s})]_G$ is completely determined by the non-equivariant invariants for X and for X/G. Among the latter, the relation*

$$[\mu(X, \mathfrak{s})] \equiv \sum_{J(G, \mathfrak{s})} [\mu(X/G, \mathfrak{s}_j)] \, mod \, p$$

is satisfied.

The comparison map, however, is not injective in general. This is proved in [39] using Adams spectral sequence calculations. To find geometrical applications for these homotopy theoretical computations looks like a challenging problem.

10. Final remarks

There is no chance to determine stable cohomotopy invariants by direct computation. This seems obvious. So the only way to get further information out of the monopole map is through a better conceptual understanding.

Any improvement in our knowledge about the groups which arise as equivariant stable cohomotopy groups in this field could help as a guideline to computing invariants as well as to constructing 4-manifolds. We know disturbingly few examples of non-vanishing refined invariants. All examples known at the moment are powers of the Hopf map η. Actually this reflects the fact that η by Pontrjiagin-Thom describes the Lie group framing of the group $\mathbb{T}$ acting.

A hypothetical way to realizing other stable cohomotopy elements was pointed out in 9.4: Construct a minimal counterexample to the $\frac{11}{8}$-conjecture! Now we know, where to start the search (9.3). It doesn't look like that hopeless an enterprise anymore.

It were symmetry considerations which lead to 9.4. Indeed, symmetry considerations may be a key to further progress. Let's dwell upon it a little more. One can consider the monopole map as a map between infinite dimensional bundles over some configuration space $Conf(X)$ consisting of all the choices made: metrics, $spin^c$-connections, harmonic 1-forms. There is a symmetry group $\mathbb{G}$ acting: It is an extension of the subgroup of the diffeomorphism group preserving the K-orientation by some gauge group. Ideally, the monopole map can be understood as an Euler class of the virtual index bundle in a "proper stable $\mathbb{G}$-equivariant cohomotopy group"

$$\pi^0_{\mathbb{G}}(Conf(X); \operatorname{ind} l)$$

with twisting in an element of "proper $\mathbb{G}$-equivariant KO-theory". The space $Conf(X)$ is the classifying space for proper $\mathbb{G}$-actions. The obvious map $E\mathbb{G} \to Conf(X)$ from the classifying space of free actions induces a "Segal map"

$$\pi^0_{\mathbb{G}}(Conf(X); \operatorname{ind} l) \to \pi^0_{\mathbb{G}}(E\mathbb{G}; \operatorname{ind} l).$$

In analogy to the compact Lie group case one would expect the latter group to be isomorphic (or at least related) to non-equivariant stable cohomotopy $\pi^0(B\mathbb{G}; \operatorname{ind} l)$. Now the classifying space $B\mathbb{G}$ of the group $\mathbb{G}$ indeed classifies parametrized families of K-oriented 4-manifolds. The image of the monopole class in this last group therefore is the universal parametrized stable cohomotopy invariant. Of course, everything here is ill defined and probably cannot be made precise at all. However, it can be made precise for compact approximations, i.e. for compact subgroups of $\mathbb{G}$ or for finite equivariant subcomplexes of $Conf(X)$. This might lead to information on the diffeomorphism groups of 4-manifolds. Already the case of the four-dimensional sphere looks interesting.

Interesting first results in this direction, relating diffeomorphisms of 4-manifolds to parametrized stable cohomotopy invariants over the 1-sphere, can be found in a recent preprint [32].

The space $Conf(X)$ might also be of interest for considering the behaviour of the stable cohomotopy invariants at its "boundary", i.e. study the behaviour of the maps under degeneration of the manifolds.

Another challenging direction of research is to find homotopy interpretations of Donaldson invariants and of Gromov-Witten invariants and to relate these concepts. At the moment these seem to be totally out of reach.

Most urgently needed, however, are more general concepts of gluing. Ideally, there should be relative invariants for manifolds with boundaries defining a "stable homotopy" quantum field theory. Sadly enough, such concepts are still missing.

A number of speculative preprints on this topic have been circulating. Just at the time of writing at least one of them is being published. The problems (cf. 7.5, 7.7) concerning well-definedness had been pointed out repeatedly to the authors as well as to the editors.

References

[1] J.F. Adams, *Prerequisites (on equivariant stable homotopy) for Carlsson's lecture*, in: Algebraic Topology (Aarhus 1982), Springer LNM 1051, Berlin (1982), 483-532.

[2] M. Atiyah, *Bott periodicity and the index of elliptic operators*, Quart. J. Math. Oxford (2) **19** (1968), 113-140.

[3] M. Atiyah, R. Bott and A. Shapiro, *Clifford modules*, Topology **3**, Suppl. 1 (1964), 3-38.

[4] W. Barth, C. Peters and A. van de Ven, *Compact complex surfaces*, Ergebnisse der Mathematik, Springer-Verlag Berlin, 1984.

[5] S. Bauer, *Diffeomorphism types of elliptic surfaces with $p_g = 1$*, J. reine angew. Math. **451** (1994), 89-148.

[6] S. Bauer, *A stable cohomotopy refinement of Seiberg-Witten invariants: II*, Invent. math. **155** (2004), 21-40.

[7] S. Bauer and M. Furuta, *A stable cohomotopy refinement of Seiberg-Witten invariants: I*, Invent. math. **155** (2004), 1-19.

[8] M.C. Crabb and K.H. Knapp, *On the codegree of negative multiples of the Hopf map*, Proc. Roy. Soc. Edinburgh, **107 A** (1987), 87-107.

[9] M.C. Crabb and K.H. Knapp, *James numbers*, Math. Ann. **282** (1988), 395-422.

[10] T. tom Dieck, *Transformation Groups.* de Gruyter, Berlin, 1987.

[11] S.K. Donaldson, *An application of gauge theory to four-dimensional topology*, Jour. Diff. Geom. **18** (1983), 279-315.

[12] S.K. Donaldson, *The orientation of Yang-Mills moduli spaces and 4-dimensional topology*, Jour. Diff. Geom. **26** (1987), 397-428.

[13] S.K. Donaldson, *Yang-Mills invariants of four-manifolds*, in: Geometry of low-dimensional manifolds, 1 (Durham 1989), London Math. Soc. Lecture Notes Series **150** (1990), 5-40.

[14] S.K. Donaldson, *The Seiberg-Witten Equations and 4-Manifold Topology*, Bull. Amer. Math. Soc. **33** (1996), 45-70.

[15] W.G. Dwyer and D.M. Kan, *An obstruction theory for diagrams of simplicial sets*, Nederl. Akad. Wetensch. Indag. Math. **46** (1984), no.2, 139-146.

[16] W.G. Dwyer and D.M. Kan, *Singular functors and realization functors*, Nederl. Akad. Wetensch. Indag. Math. **46** (1984), no.2, 147-153.

[17] N. Elkies, *A characterization of the $\mathbb{Z}^n$-lattice,* Math. Res. Lett., **2** (1995), 321-326.

[18] M. Fernández, M. Gotay and A. Gray, *Compact parallelizable 4-dimensional symplectic and complex manifolds*, Proc. Amer. Math.Soc. **103** (1988), 1209-1212.

[19] R. Fintushel and R. Stern, *Knots, links and 4-manifolds*, Invent. Math. **138** (1998), 363-400.

[20] M. Furuta, *Monopole equation and the 11/8-conjecture*, Math. Res. Lett. **8** (2001), 279-291.

[21] M. Furuta, *Finite Dimensional Approximations in Geometry*, Proceedings ICM 2002, Vol. II, 395-403.

[22] M. Furuta, Y. Kametani and H. Matsue, *Spin 4-manifolds with signature $= -32$*, Math. Res. Lett. **8** (2001), 293-301.

[23] M. Furuta, Y. Kametani, H. Matsue and N. Minami *Stable-homotopy Seiberg-Witten invariants and Pin bordisms*, Preprint (UTMS 2000-46, University of Tokyo)

[24] H.-J. Geiges, *Symplectic structures on T^2-bundles over T^2*, Duke Math. J. **67** (1992), 539-555.

[25] R.E. Gompf and A.I. Stipsicz, *4-Manifolds and Kirby Calculus.* Graduate Studies in Mathematics **20**, AMS, Providence, 1999

[26] F. Hirzebruch and H. Hopf, *Felder von Flächenelementen in 4-dimensionalen Mannig-faltigkeiten*, Math. Ann. **136** (1958), 156-172.

[27] M. Ishida and C. LeBrun, *Spin Manifolds, Einstein Metrics and Differential Topology*, Math. Res. Lett. **9** (2002), 229-240.

[28] M. Ishida and C. LeBrun, *Curvature, Connected Sums and Seiberg-Witten Theory*, Comm. An. Geom. (to appear).

[29] N.H. Kuiper, *The homotopy type of the unitary group of Hilbert space*, Topology **3** (1965), 19-30.

[30] T.J. Li and A. Liu, *General wall crossing formula*, Math. Res. Lett. **2**(1995), 797-810.

[31] R. Mandelbaum and B. Moishezon, *On the topology of simply connected algebraic surfaces*, Trans. Amer. Math. Soc. **260** (1980), 195-222.

[32] Ming Xu, 1-*Parameter Bauer-Furuta Invariant*, Preprint (2003).

[33] J. Morgan and Z. Szabo, *Homotopy K3-surfaces and mod 2 Seiberg-Witten invariants*, Math. Res. Lett. **4** (1997), 17-21.

[34] J. Morgan, Z. Szabo and C.H. Taubes, *A product formula for the Seiberg-Witten invari-ants and the generalized Thom conjecture*, Jour. Diff. Geom. **44** (1996), 706-788.

[35] B. Schmidt, *Ein Kriterium für die Existenz äquivarianter Abbildungen zwischen reellen Darstellungssphären der Gruppe Pin(2)*, Diplomarbeit, Universität Bielefeld (1997).

[36] B. Schmidt, *Spin 4-manifolds and Pin(2)-equivariant homotopy theory*, Dissertation, Bielefeld (2003). urn:nbn:hbz:361-3165, electronically available at http://bieson.ub.uni-bielefeld.de/volltexte/2003/316/

[37] A.S. Schwarz, Dokl. Akad. Nauk USSR **154**, pp.61-63. English translation: *The homotopic Topology of Banach Spaces* AMS Transl. Vol. bf 5 (1964), 57-59.

[38] G.B. Segal, *Equivariant K-theory*, Publ. Math. Inst. Hautes Études Sci. **34** (1968), 129-151.

[39] M. Szymik *Galois Symmetries of 4-Manifolds and Equivariant Stable Homotopy Theory*, Dissertation, Bielefeld (2002).

[40] C.H. Taubes, *The Seiberg-Witten invariants and symplectic forms*, Math. Res. Letters **1** (1994), 809-822.

[41] C.H. Taubes, *SW ⇒ Gr: from the Seiberg-Witten equations to pseudo-holomorphic curves*, J. Amer. Math. Soc. **9** (1996), 819-918.

[42] W. Thurston, *Some simple examples of symplectic manifolds*, Proc. Amer. Math. Soc. **55** (1976), 467-468.

[43] C.T.C. Wall, *On simply-connected 4-manifolds*, J. London Math. Soc. **39** (1964), 141-149.

[44] E. Witten, *Monopoles and four-manifolds*, Math. Res. Lett. **1**(1994), 769-796.

FAKULTÄT FÜR MATHEMATIK, UNIVERSITÄT BIELEFELD, PF 100131, D-33501 BIELEFELD

Current address: Institute for Advanced Study, Einstein Drive, Princeton, NJ 08540

E-mail address: `bauer@mathematik.uni-bielefeld.de`